More difficult questions are marked with either * or **. Those marked * are at the level which MATH2069 students will have to solve in order to be sure of getting a Credit, or to have a chance of a Distinction or High Distinction. Those marked ** are mainly intended for MATH2969 students.

1. Apply the Breadth-First Search algorithm to find a spanning tree of the Petersen graph. Explain why, no matter what choices you make, the isomorphism class of the resulting tree is always the same.

2. What sort of spanning tree do you obtain by applying the Breadth-First Search algorithm to the complete graph $K_n$? What about if you apply the Depth-First Search algorithm instead?

3. Show that if $T$ is a tree and $v, w$ are non-adjacent vertices of $T$, then $T + \{v, w\}$ contains exactly one cycle.

4. Use Prim’s Algorithm to find minimal spanning trees in these weighted graphs, starting with the vertex marked with a star. (Here the labels on the vertices have been omitted so as not to clutter the pictures.)

5. Find the number of spanning trees of each of the following graphs.

6. How many spanning trees does the graph $K_n - \{1, n\}$ have? (Hint: since Cayley’s Formula gives the number of spanning trees of $K_n$, you just need to work out how many of these contain the edge $\{1, n\}$.)
7. The adjacency matrix of a graph $G$ with vertex set $\{1, 2, \cdots, n\}$ is the $n \times n$ matrix $A$ whose entries are defined by

$$a_{ij} = \begin{cases} 
1 & \text{if } \{i, j\} \text{ is an edge of } G, \\
0 & \text{otherwise.}
\end{cases}$$

(a) Show that the $(i, j)$ entry of the $\ell$th power $A^\ell$ equals the number of walks from $i$ to $j$ in $G$ of length $\ell$.

(b) Show that $\frac{1}{2} \text{tr}(A^2)$ equals the number of edges in $G$.

(c) Show that $\frac{1}{6} \text{tr}(A^3)$ equals the number of 3-cycles in $G$.

8. Let $G$ be a connected graph. A strongly connected orientation of $G$ is a way of putting a direction on each edge (i.e. rather than just being between $v$ and $w$, it is either from $v$ to $w$ or from $w$ to $v$) so that there is a walk from any vertex to any other which obeys these directions.

(a) Find a strongly connected orientation of the following graph (indicate the directions with arrows on the edges).

(b) Prove that if $G$ has a strongly connected orientation, it contains no bridges.

**(c) Prove the converse to the previous part, as follows. Let $G$ be a connected graph with no bridges, and let $T$ be a spanning tree of $G$ produced using the Depth-First Search Algorithm, with the vertices labelled $v_1, \cdots, v_n$ in the order in which they were added to the tree. For any edge $\{v_i, v_j\}$ of $G$ where $i < j$, direct it from $v_i$ to $v_j$ if it belongs to the tree $T$, and from $v_j$ to $v_i$ if it does not. Show that this is a strongly connected orientation of $G$.

**9. Find the number of spanning trees of the complete bipartite graph $K_{p,q}$. 