keep substituting until a pattern is clear, whereupon you jump to the point at which the terms of the sequence disappear, leaving a large formula with an inevitable “…” . This is not as helpful as you might think, because it is often just rephrasing the recurrence relation in a more cumbersome way.

**Example 2.6.** Consider the sequence defined by

\[ a_0 = 1, \quad a_n = \sqrt{1 + a_{n-1}} \quad \text{for} \quad n \geq 1. \]

If you unravel the recurrence relation, you get

\[ a_n = \sqrt{1 + a_{n-1}} \]
\[ = \sqrt{1 + \sqrt{1 + a_{n-2}}} \]
\[ = \sqrt{1 + \sqrt{1 + \sqrt{1 + a_{n-3}}}} \]
\[ \vdots \]
\[ = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots + \sqrt{1 + 1}}}} \quad (\text{with} \quad n \quad \sqrt{\text{signs}}). \]

However, this way of writing \( a_n \) tells us nothing more than the recurrence relation did; it certainly doesn’t count as a solution of the recurrence relation.

However, there are cases where the unravelling procedure helps to reveal that the problem is of a kind you already know how to solve.

**Example 2.7.** The triangular number \( t_n \) is the number of dots in a triangular array with \( n \) dots on each side. Since removing one of the sides produces a smaller triangular array, this has a recursive definition of sum type:

\[ t_0 = 0, \quad t_n = t_{n-1} + n \quad \text{for} \quad n \geq 1. \]

Unravelling this recurrence relation gives the non-closed formula

\[ t_n = 0 + 1 + 2 + \cdots + (n - 1) + n \]

Summing the arithmetic progression gives the solution of the recurrence:

\[ t_n = \frac{n(n+1)}{2} \]

This sequence begins 0, 1, 3, 6, 10, 15, 21, 28, \ldots.
This is trivial, after multiplying both sides by \( r - 1 \). Of course, most people would prove (2.7) by multiplying both sides by \( r - 1 \) in the first place; but this results in a telescoping sum on the left-hand side, and so the principle of induction would be implicitly involved.

**Example 2.20.** Consider the sequence defined by

\[ a_0 = 1, \quad a_n = a_{n-1}(2^{n-1} + 1) \] for \( n \geq 1 \).

Work out the next few terms:

\[
\begin{align*}
    a_1 &= 1 \times (2^1 + 1) = 3 \\
    a_2 &= 3 \times (2^2 + 1) = 15 \\
    a_3 &= 15 \times (2^3 + 1) = 255
\end{align*}
\]

Can you guess a closed formula that fits all this data?

\[ a_n = 2^{2n} - 1 \]

To prove your formula is correct, all that remains is to check that it satisfies the recurrence relation. Write this out to make sure:

\[ 2^{2n} - 1 = (2^{2n-1} - 1)(2^{2n-1} + 1) \] for \( n \geq 1 \).

**Example 2.21.** One formula which is surprisingly easy to guess is for the sum of the first \( n \) positive integer cubes:

\[ a_n = 1^3 + 2^3 + 3^3 + \cdots + n^3. \]

Rephrasing the definition recursively:

\[ a_0 = 0, \quad a_n = a_{n-1} + n^3 \] for \( n \geq 1 \).

Working out the first few terms, we find that

\[ a_1 = 1, \quad a_2 = 9, \quad a_3 = 36, \quad a_4 = 100, \quad a_5 = 225, \]
Example 2.33. We can use Theorems 2.29 and 2.32 to solve the following second-order recurrence relation:

\[ a_0 = 2, \quad a_1 = 5, \quad a_n = 7a_{n-1} - 12a_{n-2}, \quad \text{for } n \geq 2. \]

The characteristic polynomial is \( x^2 - 7x + 12 \), which factorizes as \((x - 3)(x - 4)\); so its roots are 3 and 4. By Theorem 2.32, both the sequence \( 3^n \) and the sequence \( 4^n \) satisfy our recurrence relation (although, clearly, neither of them satisfies our initial conditions). So by Theorem 2.29, any sequence with formula \( C_13^n + C_24^n \), where \( C_1 \) and \( C_2 \) are constants (i.e. independent of \( n \)), also satisfies this recurrence relation. It so happens that we can find constants \( C_1 \) and \( C_2 \) such that the initial conditions hold also. For this we need to have the following:

\[
C_13^0 + C_24^0 = 2, \text{ i.e. } C_1 + C_2 = 2, \text{ and } \\
C_13^1 + C_24^1 = 5, \text{ i.e. } 3C_1 + 4C_2 = 5.
\]

This is a system of two linear equations in the two unknowns \( C_1 \) and \( C_2 \), and it is straightforward to find the unique solution: \( C_1 = 3, \) \( C_2 = -1. \) (For example, the second equation minus three times the first equation tells us that \( C_2 = -1 \), and then \( C_1 = 3 \) follows by substituting this in the first equation.) Since \( 3 \times 3^n + (-1) \times 4^n \) satisfies the same recurrence relation and initial conditions as \( a_n \), we must have

\[ a_n = 3 \times 3^n + (-1) \times 4^n = 3^{n+1} - 4^n. \]

So we have solved the original recurrence relation.

Example 2.34. Consider the sequence defined by

\[ a_0 = 1, \quad a_1 = 4, \quad a_n = 4a_{n-1} - 4a_{n-2} \text{ for } n \geq 2. \]

The characteristic polynomial is \( x^2 - 4x + 4 = (x - 2)^2 \) which has repeated root \( 2 \). By Theorems 2.32 and 2.29, any sequence \( a_n \) which the form \( a_n = C_12^n + C_2n2^n \) for constants \( C_1 \) and \( C_2 \) must satisfy \( a_n = 4a_{n-1} - 4a_{n-2} \). To find constants which make the initial
conditions satisfied also, we rewrite \( a_0 = 1 \) and \( a_1 = 4 \) as linear equations in \( C_1 \) and \( C_2 \):

\[
C_1 + 0 = 1 \quad \text{and} \quad 2C_1 + 2C_2 = 4
\]

The unique solution of this system of linear equations is \( C_1 = 1 \), \( C_2 = 1 \). Hence the solution of the original recurrence relation is

\[
a_n = (n + 1)2^n
\]

**Example 2.35*. Consider the sequence defined by

\[
a_0 = 2, \ a_1 = 2, \ a_n = 2a_{n-1} - 2a_{n-2} \quad \text{for} \quad n \geq 2.
\]

The characteristic polynomial is \( x^2 - 2x + 2 \), whose roots are complex: \( 1 \pm i \). Thus both \( (1 + i)^n \) and \( (1 - i)^n \) satisfy the recurrence relation, and hence so does \( C_1(1 + i)^n + C_2(1 - i)^n \) for any constants \( C_1 \) and \( C_2 \). The latter satisfies the initial conditions if and only if

\[
C_1 + C_2 = 2, \quad (1 + i)C_1 + (1 - i)C_2 = 2,
\]

which has unique solution \( C_1 = C_2 = 1 \). So we have \( a_n = (1 + i)^n + (1 - i)^n \).

We can rewrite this in a slightly better way, which removes the need for complex numbers, if we recall the polar forms \( 1 \pm i = \sqrt{2} e^{\pm i \frac{\pi}{4}} \):

\[
a_n = (1 + i)^n + (1 - i)^n = \sqrt{2}^n \left( e^{i \frac{\pi}{4}} + e^{-i \frac{\pi}{4}} \right) = 2\sqrt{2}^n \cos\left(\frac{n\pi}{4}\right).
\]

Thus the sequence \( a_n \) is almost periodic with period 8; whenever the argument of \( \cos \) recurs, the value gets multiplied by \( \sqrt{2}^8 = 16 \). If we had worked out the first few terms of the sequence, the pattern would have emerged:

\[
2, 2, 0, -4, -8, -8, 0, 16, 32, 32, 0, -64, -128, -128, \ldots
\]

In all of these examples we were able to find a linear combination of the solutions provided by Theorem 2.32 which satisfied the right initial conditions to be the sequence we were looking for. This was no coincidence, because it always happens:
Example 2.48. We can now find the general solution of the tower of Hanoi recurrence relation \( a_n = 2a_{n-1} + 1 \). The corresponding homogeneous recurrence relation \( b_n = 2b_{n-1} \) has general solution \( b_n = C2^n \), where \( C \) is a constant. Since the extra term in the non-homogeneous recurrence is also a constant (if you like, a polynomial in \( n \) of degree 0), we look for a solution \( p_n = C' \) where \( C' \) is a constant. The equation we need \( C' \) to satisfy is \( C' = 2C' + 1 \) which means that \( C' = -1 \). So by Theorem 2.45, the general solution of \( a_n = 2a_{n-1} + 1 \) is

\[
a_n = C2^n - 1 \quad \text{for some constant } C.
\]

The constant \( C \) is determined by the initial condition, i.e. the value of \( a_0 \). In the tower of Hanoi example we had \( h_0 = 0 \), so \( h_n = 2^n - 1 \) as seen before.

Example 2.49*. Recall from Example 1.82 that we have a closed formula for the elements of the \( k = 2 \) column of the Stirling triangle: \( S(n, 2) = 2^{n-1} - 1 \) for all \( n \geq 1 \). So Theorem 1.81 gives us a recurrence relation for the elements of the \( k = 3 \) column:

\[
S(1, 3) = 0, \quad S(n, 3) = 3S(n - 1, 3) + 2^{n-2} - 1 \quad \text{for } n \geq 2.
\]

This is a first-order non-homogeneous recurrence. According to Theorem 2.45, the general solution of \( a_n = 3a_{n-1} + 2^{n-2} - 1 \) has the form \( a_n = b_n + p_n \), where \( b_n \) satisfies the first-order homogeneous relation \( b_n = 3b_{n-1} \), and \( p_n \) is a particular solution of the recurrence relation \( p_n = 3p_{n-1} + 2^{n-2} - 1 \). We have \( b_n = C3^n \) for some constant \( C \), so we just need to find \( p_n \). This time the extra term is not a polynomial in \( n \), but applying the same principle of trying things of the same form as the extra term, we look for a solution of the form \( p_n = C_12^n + C_2 \) where \( C_1 \) and \( C_2 \) are further constants. We need to have the following equation for all \( n \geq 2 \):

\[
C_12^n + C_2 = 3(C_12^{n-1} + C_2) + 2^{n-2} - 1
\]

\[
= \left( \frac{3C_1}{2} + \frac{1}{4} \right) 2^n + (3C_2 - 1).
\]

For this to hold for all \( n \geq 2 \), it has to be true that \( C_1 = \frac{3C_2 + 1}{2} \) and \( C_2 = 3C_2 - 1 \). Hence \( C_1 = -\frac{1}{2} \) and \( C_2 = \frac{1}{2} \), and our particular solution is