1. Give recursive definitions of the following sequences.
   (a) The sequence of powers of 2: \(2^0 = 1\), and for \(n \geq 1\),
   \[
   2^n = 2 \times 2^{n-1}
   \]
   (b) The Catalan numbers: \(c_0 = 1\), and for \(n \geq 1\),
   \[
   c_n = c_0c_{n-1} + c_1c_{n-2} + \cdots + c_{n-2}c_1 + c_{n-1}c_0
   \]

2. The following is meant to be a proof by induction that
   \[
   \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}
   \]
   for all \(n \geq 1\), but the lines are jumbled.
   Suppose \(n \geq 2\) and the claim holds for \(n - 1\).
   \[
   = \frac{n-1}{n} + \frac{1}{n(n+1)} = \frac{(n-1)(n+1)+1}{n(n+1)}
   \]
   (1)
   The base case is true because
   (2)
   This establishes the inductive step and completes the proof.
   (3)
   Then
   \[
   \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)}
   \]
   (4)
   \[
   = \frac{1}{1 \times 2} + \frac{1}{2 \times 2} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)}
   \]
   (5)
   \[
   \frac{1}{1 \times 2} = \frac{1}{2} = \frac{1}{1+1}
   \]
   \[
   = \frac{1}{n(n+1)} = \frac{n}{n+1}
   \]
   (6)
   \[
   \frac{1}{n^2} = \frac{1}{1+1}
   \]
   (7)
   What is the correct order of the lines? 3, 6, 1, 5, 2, 7, 4
3. Consider the sequence defined by \(a_0 = 0, \ a_n = a_{n-1} + 2n\) for \(n \geq 1\).
   (a) Unravelling gives the non-closed formula
   \[a_n = 2 + 4 + 6 + \cdots + 2n\ \text{for all} \ n \geq 0.\]
   (b) Summing the arithmetic progression gives the closed formula
   \[a_n = n^2 + n\ \text{for all} \ n \geq 0.\]
   (c) Check that the formula in the previous part satisfies the recurrence relation:
   \[n^2 + n = (n - 1)^2 + n - 1 + 2n\ \text{is indeed true.}\]

4. Consider the recurrence relation \(a_n = a_{n-1} + 6a_{n-2}\), for \(n \geq 2\).
   (a) What is the characteristic polynomial? \(x^2 - x - 6\)
   (b) What are the roots of this polynomial? \(x = 3, -2\)
   (c) Write down the general solution:
   \[a_n = C_1 3^n + C_2 (-2)^n\ \text{for some constants} \ C_1, C_2.\]
   (d) Find the solution when \(a_0 = 1, \ a_1 = -1\):
   \[a_n = \frac{1}{5} 3^n + \frac{4}{5} (-2)^n\]
5. Consider the recurrence relation \( a_n = -2a_{n-1} - a_{n-2} \), for \( n \geq 2 \).

(a) What is the characteristic polynomial? \( x^2 + 2x + 1 \)

(b) What are the roots of this polynomial? \( x = -1 \) (repeated root)

(c) Write down the general solution:
\[
a_n = C_1(-1)^n + C_2n(-1)^n \text{ for some constants } C_1, C_2.
\]

(d) Find the solution when \( a_0 = 1, a_1 = -3 \):
\[
a_n = (-1)^n(2n + 1)
\]

6. Consider the recurrence relation \( a_n = -a_{n-2} \), for \( n \geq 2 \).

(a) What is the characteristic polynomial? \( x^2 + 1 \)

(b) What are the roots of this polynomial? \( x = i, -i \)

(c) Write down the general solution:
\[
a_n = C_1i^n + C_2(-i)^n \text{ for some constants } C_1, C_2.
\]

(d) If \( a_0 = 0 \) and \( a_1 = 1 \), what is \( a_7 \)?
\[
a_7 = -1
\]
7. For each of the following statements, write T for true or F for false.

(a) Apart from 0 and 1, every Fibonacci number is prime.  
F

(b) For all \( n \geq 7 \), we have \( 3^n < n! \).
T

(c) If \( a_0 = 3 \) and \( a_n = 2a_{n-1} \) for \( n \geq 1 \), then \( a_n = 3 \times 2^n \).
T

(d) \( 1 + 3 + 5 + 7 + \cdots + 99 = 99^2 \).
F

(e) \( 1^3 + 2^3 + 3^3 + \cdots + 99^3 \) is a perfect square.
T

(f) \( a_n = 3a_{n-2} \) is a second-order homogeneous linear recurrence.
T

(g) The roots of the quadratic \( x^2 - x - 1 \) are \( \pm \frac{1 + \sqrt{5}}{2} \).
F

(h) If \( a_0 = 4 \) and \( a_n = 2a_{n-1} + 1 \) for \( n \geq 1 \), then \( a_5 = 159 \).
T

(i) If \( a_0 = 2 \), \( a_1 = 0 \), and \( a_n = a_{n-2} \) for \( n \geq 2 \), then \( a_n = 1 + (-1)^n \).
T

(j) \( a_n = n^2 2^n \) is a solution of \( a_n = 4a_{n-1} - 4a_{n-2} \) (\( n \geq 2 \)).
F