More difficult questions are marked with either * or **. Those marked * are at the level which MATH2069 students will have to solve in order to be sure of getting a Credit, or to have a chance of a Distinction or High Distinction. Those marked ** are mainly intended for MATH2969 students. Some answers are at the end of the sheet.

1. Prove by induction that, for all \( n \geq 0 \),
   (a) \( n^3 + 5n \) is a multiple of 3 (i.e. \( n^3 + 5n = 3\ell \) for some integer \( \ell \)).
   (b) \( 5^n - 4n - 1 \) is a multiple of 16.

2. Use the characteristic polynomial to solve the following recurrence relations:
   (a) \( a_n = 5a_{n-1} - 6a_{n-2} \) for \( n \geq 2 \), where \( a_0 = 2 \), \( a_1 = 5 \).
   (b) \( a_n = 4a_{n-1} - 3a_{n-2} \) for \( n \geq 2 \), where \( a_0 = -1 \), \( a_1 = 2 \).
   (c) \( a_n = 4a_{n-1} - 4a_{n-2} \) for \( n \geq 2 \), where \( a_0 = 3 \), \( a_1 = 8 \).
   (d) \( a_n = 6a_{n-1} - 9a_{n-2} \) for \( n \geq 2 \), where \( a_0 = 2 \), \( a_1 = -3 \).
   *(e) \( a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \) for \( n \geq 3 \), where \( a_0 = 3 \), \( a_1 = 5 \), \( a_2 = 11 \).
   *(f) \( a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} \) for \( n \geq 3 \), where \( a_0 = 2 \), \( a_1 = 4 \), \( a_2 = 16 \).

3. Companies A and B control the market for a certain product. From one year to the next, A retains 70% of its custom and loses to B the remaining 30%, while B retains 60% of its custom and loses to A the remaining 40%. Let \( a_n \) denote the market share of company A after \( n \) years (thus, that of company B is \( 1 - a_n \)).
   (a) Write down a recurrence relation expressing \( a_n \) in terms of \( a_{n-1} \), for \( n \geq 1 \).
   (b) Solve the recurrence relation, in the sense of giving a closed formula for \( a_n \), in terms of \( a_0 \).
   (c) Hence prove that the market share of company A in the long run (i.e. the limit of \( a_n \) as \( n \to \infty \)) is independent of its initial market share \( a_0 \).

4. Let \( b_n \) be the number of ways of forming a line of \( n \) people distinguished only by whether they are male (M) or female (F), such that no two males are next to each other. For example, the possibilities with 3 people are FFF, FFM, FMF, MFF, and MFM, so \( b_3 = 5 \). Write down a recurrence relation for \( b_n \). Do you recognize the sequence?

5. Define a sequence recursively by \( a_0 = 1 \), \( a_1 = 2 \), and \( a_n = a_{n-1}a_{n-2} \) for \( n \geq 2 \).
   (a) Find \( a_2 \), \( a_3 \), \( a_4 \), \( a_5 \) and \( a_6 \).
   (b) Prove that \( a_n = 2^{F_n} \), where \( F_0, F_1, F_2, \ldots \) is the Fibonacci sequence.
6. Imagine a $2^n \times 2^n$ array of equal-sized squares, where $n$ is some positive integer. We want to cover this array with non-overlapping L-shaped tiles, each of which exactly covers three squares (one square and two of the adjacent squares, not opposite to each other). Since the number of squares is not a multiple of 3, we need to remove one square before we start. Prove by induction that no matter which square we remove, the remaining squares can be covered by these L-shaped tiles.

7. The following argument ‘proves’ that whenever a group of people is in the same room, they all have the same height. There must be an invalid step; find it.

We argue by induction on the number $n$ of people in the room. The $n = 1$ case is obviously true. Suppose that $n \geq 2$ and that the claim holds for rooms with $n - 1$ people. Let $P_1, P_2, \ldots, P_n$ be the $n$ people in this room. If $P_n$ were to leave the room we would have a room with $n - 1$ people, so by the inductive hypothesis, $P_1, P_2, \ldots, P_{n-1}$ all have the same height. We can apply the same reasoning with $P_1$ leaving the room, so $P_2, \ldots, P_{n-1}, P_n$ all have the same height. But $P_2$ is in both these collections, so all of $P_1, P_2, \ldots, P_n$ have the same height. This establishes the inductive step, and so the claim holds for all $n$ by induction.

8. For which $n$ is the Fibonacci number $F_n$ even, and for which $n$ is $F_n$ odd? Prove your answer by induction.

9. Suppose we want to solve a recurrence relation which is almost a $k$th-order homogeneous linear recurrence relation, but with an extra constant term $C$:

$$a_n = r_1a_{n-1} + r_2a_{n-2} + \cdots + r_ka_{n-k} + C,$$

for all $n \geq k$.

Let $p(x) = x^k - r_1x^{k-1} - \cdots - r_k$ be the characteristic polynomial of the homogeneous recurrence relation obtained by omitting $C$.

(a) Show that any solution $a_n$ also satisfies the $(k + 1)$th-order linear homogeneous recurrence relation with characteristic polynomial $(x - 1)p(x)$.

(b) Hence describe the general solution $a_n$ in terms of the roots of $p(x)$. (The answer will depend on whether 1 is a root of $p(x)$ or not.)

Selected answers:

2. $2^n + 3^n$, $\frac{3^{n+1} - 5}{2}$, $(n + 3)2^n$, $3^n(2 - 3n)$, $3^n + 2$, $2^n(2 - n + n^2)$. 