1. Use generating functions to solve the following recurrence relations. (Hint: you can re-use some of the calculations in last week’s Tutorial.)

(a) \( a_n = 5a_{n-1} - 6a_{n-2} \) for \( n \geq 2 \), where \( a_0 = 2, a_1 = 5 \).

**Solution:** Let \( A(z) \) denote the generating function of this sequence. We use the recurrence relation to express \( A(z) \) in terms of itself:

\[
A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots = 2 + 5z + (5a_1 - 6a_0)z^2 + (5a_2 - 6a_1)z^3 + (5a_3 - 6a_2)z^4 + \cdots
\]

\[
= 2 + 5z + 5(a_1 z^2 + a_2 z^3 + a_3 z^4 + \cdots) - 6(a_0 z^2 + a_1 z^3 + a_2 z^4 + \cdots)
\]

\[
= 2 + 5z + 5z(A(z) - 2) - 6z^2A(z).
\]

Rearranging this gives \((1 - 5z + 6z^2)A(z) = 2 - 5z\), which in turn yields

\[
A(z) = \frac{2 - 5z}{(1 - 2z)(1 - 3z)} = \frac{1}{1 - 2z} + \frac{1}{1 - 3z}.
\]

We deduce that \( a_n = 2^n + 3^n \).

(b) \( a_n = 6a_{n-1} - 9a_{n-2} \) for \( n \geq 2 \), where \( a_0 = 3, a_1 = 9 \).

**Solution:** Let \( A(z) \) denote the generating function of this sequence. We have:

\[
A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots
\]

\[
= 3 + 9z + (6a_1 - 9a_0)z^2 + (6a_2 - 9a_1)z^3 + (6a_3 - 9a_2)z^4 + \cdots
\]

\[
= 3 + 9z + 6(a_1 z^2 + a_2 z^3 + a_3 z^4 + \cdots) - 9(a_0 z^2 + a_1 z^3 + a_2 z^4 + \cdots)
\]

\[
= 3 + 9z + 6z(A(z) - 3) - 9z^2A(z).
\]

Rearranging this gives \((1 - 6z + 9z^2)A(z) = 3 - 9z\), which in turn yields

\[
A(z) = \frac{3 - 9z}{(1 - 3z)^2} = \frac{3}{1 - 3z}.
\]

We deduce that \( a_n = 3^{n+1} \).

(c) \( a_n = 3a_{n-1} + 2^{n-1} \) for \( n \geq 1 \), where \( a_0 = 0 \).

**Solution:** Let \( A(z) \) denote the generating function. We have:

\[
A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots
\]

\[
= 0 + (3a_0 + 2^0)z + (3a_1 + 2^1)z^2 + (3a_2 + 2^2)z^3 + (3a_3 + 2^3)z^4 + \cdots
\]

\[
= 3(a_0 z + a_1 z^2 + a_2 z^3 + a_3 z^4 + \cdots) + (2^0 z + 2^1 z^2 + 2^2 z^3 + \cdots)
\]

\[
= 3zA(z) + \frac{z}{1 - 2z}.
\]
Rearranging this gives

\[ A(z) = \frac{z}{(1-2z)(1-3z)}. \]

As in Q5(a) of last week’s Tutorial, we deduce that \( a_n = 3^n - 2^n \).

(d) \( a_n = -a_{n-1} + 4(-1)^{n-1} \) for \( n \geq 1 \), where \( a_0 = -1 \).

**Solution:** Let \( A(z) \) denote the generating function. We have:

\[
A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots
\]

\[
= -1 + (-a_0 + 4)z + (-a_1 - 4)z^2 + (-a_2 + 4)z^3 + (-a_3 - 4)z^4 + \cdots
\]

\[
= -1 - (a_0 z + a_1 z^2 + a_2 z^3 + \cdots) + 4(z - z^2 + z^3 - z^4 + \cdots)
\]

\[
= -1 - zA(z) + \frac{4z}{1+z},
\]

Rearranging this gives \((1+z)A(z) = \frac{3z-1}{1+z}\), which yields

\[ A(z) = \frac{3z - 1}{(1+z)^2}. \]

As in Q6(a) of last week’s Tutorial, we deduce that \( a_n = (-1)^{n-1}(4n+1) \).

*(e) \( a_n = a_{n-1} + 2n + 4 \) for \( n \geq 1 \), where \( a_0 = 5 \).

**Solution:** Let \( A(z) \) denote the generating function. We have:

\[
A(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n
\]

\[
= 5 + \sum_{n=1}^{\infty} (a_{n-1} + 2n + 4) z^n
\]

\[
= 5 + \sum_{n=1}^{\infty} a_{n-1} z^n + 2 \sum_{n=1}^{\infty} (n+1) z^n + 2 \sum_{n=1}^{\infty} z^n
\]

\[
= 5 + zA(z) + 2 \left( \frac{1}{(1-z)^2} - 1 \right) + 2 \left( \frac{1}{1-z} - 1 \right)
\]

\[
= zA(z) + 1 + \frac{2}{1-z} + \frac{2}{(1-z)^2}.
\]

Rearranging this gives

\[ A(z) = \frac{1}{1-z} + \frac{2}{(1-z)^2} + \frac{2}{(1-z)^3}. \]

As in Q5(c) of last week’s Tutorial, we deduce that \( a_n = n^2 + 5n + 5 \). (This is admittedly a roundabout way of calculating \( a_n \), which is just 5 + 6 + 8 + 10 + \cdots + (2n + 4).)

*(f) \( a_n = 2a_{n-1} - a_{n-2} + 4 \) for \( n \geq 2 \), where \( a_0 = 0, a_1 = 3 \).
Solution: Let \( A(z) \) denote the generating function. We have:

\[
A(z) = a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n
\]

\[
= 3z + \sum_{n=2}^{\infty} (2a_{n-1} - a_{n-2} + 4) z^n
\]

\[
= 3z + 2 \sum_{n=2}^{\infty} a_{n-1} z^n - \sum_{n=2}^{\infty} a_{n-2} z^n + 4 \sum_{n=2}^{\infty} z^n
\]

\[
= 3z + 2z A(z) - z^2 A(z) + 4 \left( \frac{1}{1-z} - 1 - z \right)
\]

\[
= -4 - z + 2z A(z) - z^2 A(z) + 4 \cdot \frac{1}{1-z}.
\]

Rearranging this gives \((1 - 2z + z^2)A(z) = \frac{z^2 + 3z}{1-z}\), which yields

\[
A(z) = \frac{z^2 + 3z}{(1-z)^3}.
\]

As in Q6(b) of last week’s Tutorial, we deduce that \( a_n = n(2n + 1) \).

\[*\] \( a_n = 3a_{n-1} + 4a_{n-2} - 6 \) for \( n \geq 2 \), where \( a_0 = 3, \ a_1 = 4 \).

Solution: Let \( A(z) \) denote the generating function. We have:

\[
A(z) = a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n
\]

\[
= 3 + 4z + \sum_{n=2}^{\infty} (3a_{n-1} + 4a_{n-2} - 6) z^n
\]

\[
= 3 + 4z + 3 \sum_{n=2}^{\infty} a_{n-1} z^n + 4 \sum_{n=2}^{\infty} a_{n-2} z^n - 6 \sum_{n=2}^{\infty} z^n
\]

\[
= 3 + 4z + 3z A(z) - 3 + 4z^2 A(z) - 6 \left( \frac{1}{1-z} - 1 - z \right)
\]

\[
= 9 + z + 3z A(z) + 4z^2 A(z) - \frac{6}{1-z}.
\]

Rearranging this gives

\[
A(z) = \frac{3 - 8z - z^2}{(1-4z)(1+z)(1-z)}.
\]

As in Q5(d) of last week’s Tutorial, we deduce that \( a_n = 4^n + (-1)^n + 1 \).
2. As seen in lectures, if \( A(z) \) is the generating function of the sequence \( a_0, a_1, a_2, \ldots \), then \( B(z) = \frac{A(z)}{1-z} \) is the generating function of the sequence \( b_0, b_1, b_2, \ldots \) where \( b_n = a_0 + a_1 + \cdots + a_n \). Use this to find closed formulas for:

(a) \( \binom{0}{2} + \binom{1}{2} + \cdots + \binom{n}{2} \)

**Solution:** We first need the generating function of \( \binom{n}{2} \):

\[
\sum_{n=0}^{\infty} \binom{n}{2} z^n = z^2 \sum_{n=0}^{\infty} \binom{n+2}{2} z^n = \frac{z^2}{(1-z)^3}.
\]

Hence the generating function of \( \binom{0}{2} + \binom{1}{2} + \cdots + \binom{n}{2} \) is

\[
\frac{z^2}{(1-z)^4} = z^2 \sum_{n=0}^{\infty} \binom{n+3}{3} z^n = \sum_{n=0}^{\infty} \binom{n+1}{3} z^n,
\]

and it follows that \( \binom{0}{2} + \binom{1}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3} \).

(b) \( \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k} \), for any fixed \( k \)

**Solution:** We first need the generating function of \( \binom{n}{k} \):

\[
\sum_{n=0}^{\infty} \binom{n}{k} z^n = z^k \sum_{n=0}^{\infty} \binom{n+k}{k} z^n = \frac{z^k}{(1-z)^{k+1}}.
\]

Hence the generating function of \( \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k} \) is

\[
\frac{z^k}{(1-z)^{k+2}} = z^k \sum_{n=0}^{\infty} \binom{n+k+1}{k+1} z^n = \sum_{n=0}^{\infty} \binom{n+1}{k+1} z^n,
\]

and it follows that \( \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \). (This equality can also be proved easily by induction on \( n \), using Pascal’s recurrence relation.)

(c) \( \sum_{m=0}^{n} (2m+3) 3^m \)

**Solution:** We first calculate that

\[
\sum_{n=0}^{\infty} (2n+3) 3^n z^n = 2 \sum_{n=0}^{\infty} (n+1) 3^n z^n + \sum_{n=0}^{\infty} 3^n z^n
\]

\[
= \frac{2}{1-3z} + \frac{1}{1-3z}
\]

\[
= \frac{3-3z}{(1-3z)^2}.
\]

It follows that the generating function of \( \sum_{m=0}^{n} (2m+3) 3^m \) is

\[
\frac{3-3z}{(1-z)(1-3z)^2} = \frac{3}{(1-3z)^2} = 3 \sum_{n=0}^{\infty} (n+1) 3^n z^n,
\]

so \( \sum_{m=0}^{n} (2m+3) 3^m = (n+1)3^{n+1} \).
3. For each of the sequences defined by the following recurrence relations, find the generating function \( A(z) \) by considering its derivative \( A'(z) \).

(a) \( a_n = \frac{1}{n}a_{n-2} \) for \( n \geq 2 \), where \( a_0 = 1, \ a_1 = 0 \).

**Solution:** We have

\[
A'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1} z^n = a_1 + \sum_{n=1}^{\infty} a_{n-1} z^n = zA(z).
\]

We can solve \( A'(z) = zA(z) \) using the result in lectures:

\[
A(z) = a_0 \exp\left(\frac{1}{2}z^2\right) = \exp\left(\frac{1}{2}z^2\right).
\]

In this case it is possible to find a formula for the coefficient of \( z^n \):

\[
a_n = \begin{cases} \\
\frac{1}{2^{n/2}(n/2)!}, & \text{if } n \text{ is even,} \\
0, & \text{if } n \text{ is odd.}
\end{cases}
\]

Actually, this can be deduced easily from the recurrence relation.

(b) \( a_n = \frac{1}{n} \sum_{m=0}^{n-1} m! a_{n-m-1} \) for \( n \geq 2 \), where \( a_0 = a_1 = 1 \).

**Solution:** We have

\[
A'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1} z^n = a_1 + \sum_{n=1}^{\infty} \left( \sum_{m=0}^{n} \frac{1}{m!} a_{n-m} \right) z^n = \exp(z)A(z).
\]

We can solve \( A'(z) = \exp(z)A(z) \) using the result in lectures: the ‘integral’ of \( \exp(z) \) with zero constant term is \( \exp(z) - 1 \), so \( A(z) = \exp(\exp(z) - 1) \). If you compare the recurrence relation in this question with that in Q7 of Tutorial 2, you can easily prove that \( a_n = \frac{B(n)}{n!} \) where \( B(n) \) is the Bell number defined there. So we have just shown that the exponential generating function of the sequence of Bell numbers is \( \exp(\exp(z) - 1) \).

(c) \( a_n = \frac{1}{n} \sum_{m=0}^{n-1} a_m a_{n-m-1} \) for \( n \geq 1 \), where \( a_0 = 1 \).

**Solution:** We have

\[
A'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1} z^n = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} a_m a_{n-m} \right) z^n = A(z)^2.
\]

Now \( A'(z) = A(z)^2 \) is not the kind of differential equation we have a general method for, but we have seen a formal power series whose derivative equals its square, namely \( \frac{1}{1-z} \). At this point realization dawns: the solution of the recurrence relation is simply \( a_n = 1 \) for all \( n \), so indeed \( A(z) = \frac{1}{1-z} \).

4. Define a formal power series \( \log(1 + z) \) by

\[
\log(1 + z) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n = z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \frac{1}{4} z^4 + \cdots.
\]
(a) Show that the derivative of \( \log(1 + z) \) is \( \frac{1}{1 + z} \).

**Solution:** By definition, the derivative of \( \log(1 + z) \) is

\[
\sum_{n=0}^{\infty} (n + 1) \frac{(-1)^n}{n + 1} z^n = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1 + z}.
\]

*(b)* Hence show that \( \exp(\log(1 + z)) = 1 + z \).

**Solution:** If we let \( A(z) = 1 + z \), then we obviously have

\[
A'(z) = 1 = \frac{1}{1 + z} A(z).
\]

By the result in lectures and part (a), the unique solution of \( A'(z) = \frac{1}{1 + z} A(z) \) with constant term 1 is \( A(z) = \exp(\log(1 + z)) \). Hence \( \exp(\log(1 + z)) = 1 + z \).

5. Let \( H_k(z) = \sum_{n=0}^{\infty} S(n + k, n + 1) z^n \) be the generating function of the \( k \)th diagonal of the Stirling triangle, for all \( k \geq 1 \).

(a) Prove that \( H_1(z) = \frac{1}{1 - z} \).

**Solution:** We have

\[
H_1(z) = \sum_{n=0}^{\infty} S(n + 1, n + 1) z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.
\]

(b) Prove that \( H_2(z) = \frac{1}{(1 - z)^3} \).

**Solution:** We have

\[
H_2(z) = \sum_{n=0}^{\infty} S(n + 2, n + 1) z^n = \sum_{n=0}^{\infty} \binom{n + 2}{2} z^n = \frac{1}{(1 - z)^3}.
\]

**(c)** Using the recurrence relation for the Stirling numbers, prove that

\[
H_k(z) = \frac{H_{k-1}(z) + zH'_{k-1}(z)}{1 - z}, \text{ for all } k \geq 2.
\]
**Solution:** For any $k \geq 2$, we have

$$H_k(z) = \sum_{n=0}^{\infty} S(n+k, n+1) z^n$$

$$= \sum_{n=0}^{\infty} (S(n+k-1, n) + (n+1)S(n+k-1, n+1)) z^n$$

$$= \sum_{n=1}^{\infty} S(n+k-1, n) z^n + \sum_{n=0}^{\infty} S(n+k-1, n+1) z^n$$

$$+ \sum_{n=1}^{\infty} nS(n+k-1, n+1) z^n$$

$$= z \sum_{n=0}^{\infty} S(n+k, n+1) z^n + \sum_{n=0}^{\infty} S(n+k-1, n+1) z^n$$

$$+ z \sum_{n=0}^{\infty} (n+1)S(n+k, n+2) z^n$$

$$= zH_k(z) + H_{k-1}(z) + zH'_{k-1}(z),$$

which rearranges to the claimed equation.

**(d)** Hence compute $H_3(z)$, and deduce a formula for $S(n+3, n+1)$.

**Solution:** Combining the previous two parts, we have

$$H_3(z) = \frac{1}{1-z} + \frac{3z}{(1-z)^4}$$

$$= \frac{1 + 2z}{(1-z)^5}$$

$$= \sum_{n=0}^{\infty} \binom{n+4}{4} z^n + 2 \sum_{n=1}^{\infty} \binom{n+3}{4} z^n.$$

So for all $n \geq 0$,

$$S(n+3, n+1) = \binom{n+4}{4} + 2 \binom{n+3}{4}$$

$$= \frac{(n+3)(n+2)(n+1)((n+4) + 2n)}{24}$$

$$= \frac{(3n+4)(n+3)(n+2)(n+1)}{24}.$$

**6.** For any $k \geq 1$, let $F_k(z)$ denote the exponential generating function of the $k$th column of the Stirling triangle:

$$F_k(z) = \sum_{n=k}^{\infty} \frac{S(n, k)}{n!} z^n.$$

Note that $F_1(z) = \exp(z) - 1$.  

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(a) Using the recurrence relation for the Stirling numbers, prove that
\[ F'_k(z) = F_{k-1}(z) + kF_k(z), \text{ for all } k \geq 2. \]

**Solution:** By definition of differentiation, we have
\[
F'_k(z) = \sum_{n=k-1}^{\infty} (n+1) \frac{S(n+1, k)}{(n+1)!} z^n
= \sum_{n=k-1}^{\infty} \frac{S(n+1, k)}{n!} z^n
= \sum_{n=k-1}^{\infty} \frac{S(n, k-1) + kS(n, k)}{n!} z^n
= \sum_{n=k-1}^{\infty} \frac{S(n, k-1)}{n!} z^n + \sum_{n=k}^{\infty} \frac{S(n, k)}{n!} z^n
= F_{k-1}(z) + kF_k(z),
\]
as required.

(b) Hence prove by induction that
\[ F_k(z) = \frac{1}{k!}(\exp(z) - 1)^k, \text{ for all } k \geq 1. \]

**Solution:** The \( k = 1 \) case we already know, so assume that \( k \geq 2 \) and that the result is known for \( k - 1 \). Then by the previous part,
\[
F'_k(z) = \frac{1}{(k-1)!}(\exp(z) - 1)^{k-1} + kF_k(z),
\]
which is a differential equation for \( F_k(z) \), although not of the kind we saw how to solve. Let \( G(z) = \frac{1}{k!}(\exp(z) - 1)^k \). We want to show \( F_k(z) = G(z) \), so we first check that \( G(z) \) satisfies the same differential equation as \( F_k(z) \):
\[
G'(z) = \frac{k}{k!}(\exp(z) - 1)^{k-1} \exp(z)
= \frac{k}{k!}(\exp(z) - 1)^{k-1}(1 + (\exp(z) - 1))
= \frac{1}{(k-1)!}(\exp(z) - 1)^{k-1} + kG(z).
\]
This means that if we let \( H(z) = F_k(z) - G(z) \), then
\[
H'(z) = F'_k(z) - G'(z)
= \left( \frac{1}{(k-1)!}(\exp(z) - 1)^{k-1} + kF_k(z) \right)
- \left( \frac{1}{(k-1)!}(\exp(z) - 1)^{k-1} + kG(z) \right)
= k(F_k(z) - G(z)) = kH(z),
\]
so \( H(z) \) satisfies a differential equation of the type we do know how to solve: the solution is \( H(z) = C \exp(kz) \) for some constant \( C \), which we can determine since it is the constant term of \( H(z) \). Since \( F_k(z) \) and \( G(z) \) both have zero constant term, the same is true of \( H(z) \), so \( C = 0 \) which means that \( H(z) = 0 \). So \( F_k(z) = G(z) \), completing the induction step.
(c) Show that the result of the previous part can also be deduced from the formula in Q8 of Tutorial 2.

**Solution:** We know that $k! S(n, k)$ equals the number of surjective functions $f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, k\}$, for which a formula was given in Q8 of Tutorial 2. Using this, we see that

$$F_k(z) = \sum_{n=k}^{\infty} \frac{S(n, k)}{n!} z^n$$

$$= \frac{1}{k!} \left( \sum_{n=k}^{\infty} \frac{1}{n!} \sum_{n_1, n_2, \cdots, n_k \geq 1}^{n_1 + n_2 + \cdots + n_k = n} \binom{n}{n_1, n_2, \cdots, n_k} z^n \right)$$

$$= \frac{1}{k!} \left( \sum_{n=k}^{\infty} \frac{1}{n_1! n_2! \cdots n_k!} \sum_{n_1, n_2, \cdots, n_k \geq 1}^{n_1 + n_2 + \cdots + n_k = n} z^{n_1 + n_2 + \cdots + n_k} \right)$$

$$= \frac{1}{k!} \left( \sum_{n_1, n_2, \cdots, n_k \geq 1}^{n_1 + n_2 + \cdots + n_k = n} \frac{1}{n_1! n_2! \cdots n_k!} z^{n_1 + n_2 + \cdots + n_k} \right)$$

$$= \frac{1}{k!} \left( \sum_{n_1, n_2, \cdots, n_k \geq 1}^{n_1 + n_2 + \cdots + n_k = n} \frac{z^{n_1} z^{n_2} \cdots z^{n_k}}{n_1! n_2! \cdots n_k!} \right)$$

$$= \frac{1}{k!} \left( \sum_{n_1=1}^{\infty} \frac{z^{n_1}}{n_1!} \right) \left( \sum_{n_2=1}^{\infty} \frac{z^{n_2}}{n_2!} \right) \cdots \left( \sum_{n_k=1}^{\infty} \frac{z^{n_k}}{n_k!} \right)$$

$$= \frac{1}{k!} (\exp(z) - 1)^k,$$

as required.