Formulas

You may freely use the following formulas.

**Portfolio Equations**

Let $R = \text{random return on investment in } n\text{-assets over a single period}$. Then the portfolio expected return and variance of return are

\[
\mu = \mathbb{E}\{R\} = \sum_{i=1}^{n} r_i x_i = r^T x, \quad \mathbb{E}\{R_i\} = r_i
\]

\[
\sigma^2 = \mathbb{V}\{R\} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i S_{ij} x_j = x^T S x, \quad S_{ij} = \text{cov}\{R_i, R_j\}.
\]

**Constraints**

Budget constraint: $\sum x_i = e^T x = 1$.

Asset constraints: $L_i \leq x_i \leq U_i$; e.g. $x_i \geq 0$ for no short-selling.

**Unrestricted $n$-Asset Portfolios**

Minimise $Z = -tr^T x + \frac{1}{2} x^T S x$, s.t. $e^T x = 1$.

Let $a = e^T S^{-1} e$, $b = e^T S^{-1} r$, $c = r^T S^{-1} r$, $d = ac - b^2$ ($a$, $c$, $d$ are always positive).

Then for a risk-aversion parameter $t$ the optimal solution is

\[
x = \alpha + \beta t, \quad \alpha = \frac{1}{a} S^{-1} e, \quad \beta = S^{-1}(r - \frac{b}{a} e), \quad \mu = \frac{b + dt}{a}, \quad \sigma^2 = 1 + \frac{dt^2}{a}.
\]

**Restricted $n$-Asset Portfolios**

Minimise $Z = -tr^T x + \frac{1}{2} x^T S x$ s.t. $e^T x = 1$ and $L \leq x \leq U$.

Lagrangian: $L = \frac{1}{2} x^T S x - tr^T x - \lambda(e^T x - 1) - \ell^T (x - L) - u^T (U - x)$

KT conditions: $S x - tr - \lambda e - \ell + u = 0$, $e^T x = 1$, $L \leq x \leq U$, $\ell_i(x_i - L_i) = 0$ with $\ell_i \geq 0$, $u_i(U_i - x_i) = 0$ with $u_i \geq 0$.

**Inclusion of a Riskless Asset — CAPM**

Add a riskless asset with return $r_0$. Let $\bar{r} = r - r_0 e$. The optimal solution for a risk-aversion parameter $t$ is

\[
x = tS^{-1} r \text{ (risky assets), } x_0 = 1 - e^T x \text{ (riskless asset), } \mu = r_0 + \bar{\tau} t, \quad \sigma^2 = \bar{\tau} t^2
\]

with $\bar{\tau} = r^T S^{-1} r$. The new efficient frontier is called the capital market line.

**Basic CAPM Formula**

\[
\mu = \mathbb{E}\{R\} = r_0 + \beta(\mathbb{E}\{R_M\} - r_0) \text{ with } \beta = \text{cov}\{R, R_M\}/\sigma^2 M \text{ and } M \text{ refers to the market portfolio and } \beta \text{ is a new measure of risk. The straight line relating } \mu \text{ and } \beta \text{ is called the security market line.}
\]

This is the last page of the question paper.