Portfolio problem
Samuelson 1969

$X_0 > 0$ wealth at time 0, $t_0 = x$

Owe $c_n \leq X_n$ invests

$X_n - c_n$

with return $r$

$\theta_n (X_n - c_n)$ invested into riskless bond

$(1 - \theta_n) (X_n - c_n)$ invested into share

with return $1 + R_n$

$R_n$ random, independent, identically distributed

$R_n \sim -1$

\[ Cu \quad \text{consumed} \]

\[ X_n \leftarrow (X_n - c_n) \theta_n \]

\[ U(1 - \theta_n) (X_n - c_n) \]
At time \((t+1)\),

\[
X_{t+1} = \left[ b_n (1+\r) + (1-b_n) (1+R_n) \right] \left[ (X_t - c_n) \right] \\
= \left[ 1 + b_n \r + (1-b_n) R_n \right] (X_t - c_n)
\]

\(R_n\) independent of \(R_{n-1}\), \(R_{n-2}\)
hence independent of \(X_0, \ldots, X_{t-1}\)

Find optimal

1) Consumption policy \((c_0, \ldots, c_{N-1}) = C_N\)

2) Investment policy \((b_0, \ldots, b_{N-1}) = B_N\)

We need utility to maximize conflicting aims:

- Consume as much as possible
- Obtain as much future wealth as possible at the end

Choose utility \(U(C, X)\)

\[
\max_{C, B} U(C, X)
\]

Too general
\[
U(C_n, X_n) = \sum_{n=0}^{N-1} U_n(C_n) + U_N(X_n)
\]

Different attitudes to consumption:
- Consume less now, more later
- Consume as much as possible as soon as possible
- Consume evenly all the time

Different attitudes to final wealth:
- Important, not important
  \[
  U_n(C_n) = g^n U(C_n)
  \]
  \[
  U_N(X_n) = \gamma g^N U(X_n)
  \]
- Consumption

\[
U(C_n, X_n) = \sum_{n=0}^{N-1} g^n U(C_n) + \gamma g^N U(X_n)
\]

Choose standard \( U \)

\[\alpha > 0 \quad U(x) = x^\alpha \quad 0 < \alpha < 1\]

\( U \) - strictly concave, increasing
Find

\[ V_0(x) = \max_{\alpha, \beta} \mathbb{E} \left[ \sum_{n=0}^{N-1} g^n c_n + \alpha g^n x_n \right] \]

\[ 0 < x < 1 \quad \alpha, \beta > 0 \]

\[ V_N(x) = x^\alpha \]

\[ V_{N-1}(x) = \max_{c, b} \mathbb{E} \left[ c^\alpha + \beta g \mathbb{E} V_N(x) \mid X_N = x \right] \]

\[ = \max_{c, b} \left[ c^\alpha + \beta g \mathbb{E} \left( X_N^\alpha \mid X_{N-1} = x \right) \right] \]

\[ X_N = \left[ b \left( 1 + r \right) + (1-b)(1-R) \right] (x-c) \]

\[ V_{N-1}(x) = \max_{c, b} \left[ c^\alpha + (x-c)^\alpha \mathbb{E} \left[ (1+r)b + (1-b)(1-R) \right] \right] \]

Maximize first

\[ \theta(b) = \mathbb{E} \left[ (1+r)b + (1-b)(1-R) \right] \]

let \( \theta \) be the maximizer

Exercise \( \theta''(b) < 0 \)
\[ \hat{\lambda} = \max_{\theta \in \Theta} \mathbb{E} \left[ (1 + \theta) \lambda + (1 - \theta) (R + 1) \right] \]
\[ \hat{\theta} = \max_{\theta \in \Theta} \mathbb{E} \]

Let
\[ F(t) = \left[ 1 + \left( \frac{\hat{\theta}}{\hat{\lambda}} \right)^{\frac{1}{\alpha - 1}} \right]^{1 - \alpha} \]
\[ \hat{G}(t) = \frac{1}{1 + (\hat{\theta} - \hat{\lambda}) t^{1 - \alpha}} \]

Let
\[ r(x) = \frac{1}{x} \int A F(x) \] 
\[ A(x) = F(t) x \]
\[ V_{N-K}(x) = g_{N-K} A^{(K)}(x) \]
\[ \hat{c}_{N-K}(x) = \hat{G}(F^{(K-1)}(t)) x \]
\[ F(t) = t \]

Then
\[ (\hat{c}_0(x_0), \hat{\theta}), (\hat{c}_1(x_1), \hat{\theta}), \ldots, (\hat{c}_{N-1}(x_{N-1}), \hat{\theta}) \] are optimal

\[ V_0(x) = F(x) x \]
Where to park your car?

\[ \text{Parking places} \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad \ldots \]

\[ \text{Desired location} \]

\[ n = \ldots, -2, -1, 0, 1, 2, \ldots \]

Each spot is free with probability \( p \).

If we park at \( n \)-th spot then the loss is \( |n| \).

\[ \begin{align*}
& \text{Independent} \\
& \text{Probability:} \\
& \quad P(Y_n = 1) = p \\
& \quad P(Y_n = 0) = 1 - p \\
\end{align*} \]

\[ X_n = \begin{cases} 
-\infty & \text{if } Y_n = 0 \\
-|n| & \text{if } Y_n = 1 
\end{cases} \]

\( X_n \): Loss if parking (or not) at \( n \).

Stopping rules: enough to consider \( \sum \).

\( \exists \): Stop at first \( n \geq 1 \) such that \( Y_n = 1 \).

Use \( \exists \) if \( \exists > 0 \).
Take any $z$ and define
\[
\widetilde{z} = \begin{cases} \frac{z}{2} & \text{if } z \leq 0 \\ 2 & \text{if } z > 0 \end{cases}
\]
Example

$X_0, X_1, \ldots, X_N$ independent, the same distribution

$P(a < X_k < b) = b - a$  \hspace{1cm} 0 \leq a < b \leq 1$

At any time $n \leq N$ you may accept $X_n$ as a reward.

Maximize $\mathbb{E} X_n^\nu$ over all stopping times $\tau \leq N$.

At time $N$ you get $X_N$. At time $n < N$ accept $X_n$ if $X_n > \mathbb{E}_{\text{max}} X_k = m_{n-n}^{k=n}$

$X_n = x \Rightarrow$ accept $X_n$ if $m_{n-n}^{k=n} > x$

$m_{n-n}^{k=n} = \mathbb{E}_{\text{max}} (X_{n+1}, \ldots, X_N)$

Optimal stopping time:

$\hat{\tau} = \min \{ k \leq N; X_k \geq m_{n-k}^{n-k+1} \}$

$m_{n-n}^{n-k} = \frac{n}{n+1}$
\[ 1 - \sum_{k=1}^{N^*} \frac{1}{r^k} < 0 \]

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\[ \sum_{k=1}^{N^*} \frac{1}{r^k} < 1 < \sum_{k=1}^{N^*} \frac{1}{r^k} \]

\[ \frac{dy}{y} = 1 < \int_{r^*}^{r} \frac{dy}{y} \]

\[ \log \left( \frac{N-1}{r^*} \right) = \log e \]

\[ \frac{N-1}{r^*} \approx e \]

\[ r^* \approx \frac{N}{e - \frac{1}{e}} \approx \frac{N}{e} \]
Put pricing

Don't exercise

$V_0(4) = 1.36$

$V_1(8) = 0.40$

$V_2(16) = 0$

$V_2(4) = 1$

$V_2(1) = 4$

Discounted put prices

$r = 2.5$

$r = 0$

$\mathbb{E} = 1.36$

$\mathbb{D}$

$\frac{4}{5} V_1(8) = 0.32$

$(\frac{4}{5})^2 V_2(16) = 0$

$\mathbb{D}$

$\frac{3}{5} V_1(4) = 0.40$

$(\frac{3}{5})^2 V_2(8) = 2.56$

$\mathbb{E}$

$\frac{2}{5} V_1(2) = 0.20$

$(\frac{2}{5})^2 V_2(1) = 2.56$

$\mathbb{D}$

$\frac{1}{5} V_1(1) = 0.10$

$(\frac{1}{5})^2 V_2(0) = 0.04$

$\mathbb{E}$

$\sum = 0, 1, 2, \infty$

$\tau - stopping time: stop and exercise$
Another decision rule

\[ S = 1 \]
\[ g(uu) = g(dd) = 0 \]
\[ g(du) = 1 \]
\[ S_1 (d) = 2 \]
\[ S_2 (dd) = 1 \]
\[ g(TT) = 2 \]

\[ g(uu) = 0 \quad g(ud) = 0 \quad g(du) = 1 \]
\[ g(dd) = 2 \]

Insider knowledge (I know the future!)

\[ y_n = \left( \frac{y}{5} \right)^n v_n \quad n = 0, 1, 2 \]

\[ y_0 = v_0 (s_0) \quad y_1 = v_1 (s_1) \frac{y}{5} \quad y_2 = \left( \frac{y}{5} \right)^2 v_2 (s_2) \]

\[ \mathbb{E} (y_1 | s_0) = y_0 \quad \mathbb{E} (y_2 | s_0, s_1) = \mathbb{E} (y_2 | s_1) = y_1 \]

\[ \mathbb{E} \left( y_2 | s_0, s_1 \right) = \left( \frac{y}{5} \right)^2 \mathbb{E} \left[ \frac{3}{5} s_2 \right] + \frac{1}{5} \mathbb{E} \left[ \frac{3}{5} (s_2 - s_0) + \frac{3}{5} (s_2 - s_0) \right] \]

\[ = \left( \frac{y}{5} \right)^2 \mathbb{E} \left[ \frac{3}{5} s_2 \right] \quad \mathbb{E} (y_1 | s_1) = y_1 \]

\[ \mathbb{E} (y_2 | y_1) = y_1 \quad \mathbb{E} (y_1 | y_0) = y_0 \]
Burglar

Each day he starts a house daily gain $X_1$, $X_2$, ... random independent

$$P(X_i = a_i) = P(X_i = b_i) = \frac{1}{2}$$

$$E X_i = \frac{a_i + b_i}{2}$$

Each day he risks being caught

$$P(C_n = 1) = p$$

$$P(C_n = 0) = 1 - p$$

$$0 < p < 1$$

Burglar gain

$$V_n = C_1 \ldots C_n (X_1 + \ldots + X_n)$$

When to stop? $\Sigma$ - stopping time

$$\Sigma : 1, 2, 3, \ldots$$

If hiring an employee for choosing the best candidate

$N$ candidates, you want the best

If you do not choose, the person is lost