Formulas

You may freely use the following formulas.

Portfolio Equations

Let R = random return on investment in n-assets over a single period. Then the portfolio expected return and variance of return are

$$\mu = \mathbb{E}\{R\} = \sum_{i=1}^{n} r_i x_i = \mathbf{r}^T\mathbf{x}\,, \qquad \mathbb{E}\{R_i\} = r_i$$

$$\sigma^2 = \mathbb{V}\{R\} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i S_{ij} x_j = \mathbf{x}^T\mathbf{S}\mathbf{x}\,, \qquad S_{ij} = \mathrm{cov}\{R_i, R_j\}\,.$$

Constraints

Budget constraint: $\sum_i x_i = \mathbf{e}^T\mathbf{x} = 1.$

Asset constraints: $L_i \le x_i \le U_i$; e.g. $x_i \ge 0$ for no short-selling.

Unrestricted n-Asset Portfolios

Minimise $Z = -t\mathbf{r}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x}$, s.t. $\mathbf{e}^T\mathbf{x} = 1$.
Let $a = \mathbf{e}^T\mathbf{S}^{-1}\mathbf{e}$, $b = \mathbf{e}^T\mathbf{S}^{-1}\mathbf{r}$, $c = \mathbf{r}^T\mathbf{S}^{-1}\mathbf{r}$, $d = ac - b^2$ (a, c, d are always positive).
Then for a risk-aversion parameter t the optimal solution is

$$\mathbf{x} = \boldsymbol{\alpha} + \boldsymbol{\beta} t\,, \quad \boldsymbol{\alpha} = \frac{1}{a}\mathbf{S}^{-1}\mathbf{e}\,, \quad \boldsymbol{\beta} = \mathbf{S}^{-1}(\mathbf{r} - \frac{b}{a}\mathbf{e})\,, \qquad \mu = \frac{b + dt}{a}\,, \quad \sigma^2 = \frac{1 + dt^2}{a}\,.$$

Restricted n-Asset Portfolios

Minimise $Z = -t\mathbf{r}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x}$ s.t. $\mathbf{e}^T\mathbf{x} = 1$ and $\mathbf{L} \le \mathbf{x} \le \mathbf{U}$.
Lagrangian: $L = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} - t\mathbf{r}^T\mathbf{x} - \lambda(\mathbf{e}^T\mathbf{x} - 1) - \boldsymbol{\ell}^T(\mathbf{x} - \mathbf{L}) - \mathbf{u}^T(\mathbf{U} - \mathbf{x})$
KT conditions: $\mathbf{S}\mathbf{x} - t\mathbf{r} - \lambda\mathbf{e} - \boldsymbol{\ell} + \mathbf{u} = \mathbf{0}$, $\mathbf{e}^T\mathbf{x} = 1$, $\mathbf{L} \le \mathbf{x} \le \mathbf{U}$, $\ell_i(x_i - L_i) = 0$ with $\ell_i \ge 0$, $u_i(U_i - x_i) = 0$ with $u_i \ge 0$.

Inclusion of a Riskless Asset — CAPM

Add a riskless asset with return r_0. Let $\overline{\mathbf{r}} = \mathbf{r} - r_0\mathbf{e}$. The optimal solution for a risk-aversion parameter t is

$$\mathbf{x} = t\mathbf{S}^{-1}\overline{\mathbf{r}} \text{ (risky assets)},\ x_0 = 1 - \mathbf{e}^T\mathbf{x} \text{ (riskless asset)},\ \mu = r_0 + \overline{c}t\,,\ \sigma^2 = \overline{c}t^2$$

with $\overline{c} = \overline{\mathbf{r}}^T\mathbf{S}^{-1}\overline{\mathbf{r}}$. The new efficient frontier is called the capital market line.

Basic CAPM Formula

$\mu = \mathbb{E}\{R\} = r_0 + \beta(\mathbb{E}\{R_M\} - r_0)$ with $\beta = \mathrm{cov}\{R, R_M\}/\sigma_M^2$ and M refers to the market portfolio and β is a new measure of risk. The straight line relating μ and β is called the security market line.