Concave functions and the Jensen Inequality

Definition. Let $-\infty < a < b < \infty$. We say that a function $f: (a, b) \rightarrow \mathbb{R}$ is concave if for any $x_0$, where $a < x_0 < b$ there exists a real $m$ such that for all $x$ in $(a, b)$

$$f(x_0) + m(x - x_0) \leq f(x)$$
Jensen's inequality

Assume that \( U \) is a concave function on \((a, b)\), a random variable \( X \) takes values in \((a, b)\) and \( \mathbb{E}|X| < \infty \). Then

\[
\mathbb{E} U(X) \leq U(\mathbb{E}X)
\]

Proof. In the definition of convex function put \( x_0 = \mathbb{E}X \), \( \alpha = X \).

Then for a certain \( m \)

\[
U(\mathbb{E}X) + m(X - \mathbb{E}X) \geq U(X)
\]

Apply expectation to both sides of this inequality, use the fact that

\[
\mathbb{E}(X - \mathbb{E}X) = 0
\]

to get

\[
U(\mathbb{E}X) \geq \mathbb{E}U(X)
\]
We will consider portfolio optimisation under the liquidity assumption: arbitrary quantity of any asset can be bought or sold on demand. Short-selling will be permitted: you can borrow the stock (or cash), sell it, use cash for another investment.

Variance of returns on portfolio is the measure of risk.
Example

$V_0 = 100$ initial capital

$R \left\langle \begin{array}{c}
\frac{9}{20} \\
-1
\end{array} \right\rangle$

At time $t=1$

$V_1 = (1 + R) V_0$

$\therefore V_1 = 2 V_0 + (0) \cdot V_0 = 200$

Would you $100$ for such a contract?

It depend on your attitude towards risk.
More complicated portfolio: Two stocks

Time \( t = 0 \)

\[ S_1(0) = 30 \] \[ S_2(0) = 40 \] \[ v(0) = 1,000 \] initial amount

Strategy:
Buy \( x_1 = 20 \) shares of \( S_1 \) and \( x_2 = 10 \) shares of stock \( S_2 \)
\[ 20 \times 30 + 10 \times 40 = 1000 \]

Weights:
\[ w_1 = \frac{30 \times 20}{1000} = 60\% \]
\[ w_2 = \frac{10 \times 40}{1000} = 40\% \]

Time \( t = 1 \)

\[ S_1(1) = 35 \]
\[ S_2(1) = 39 \]
\[ v(1) = 20 \times 35 + 10 \times 39 = 1,030 \]

New weights
\[ \frac{20 \times 35}{1030} \approx 64.22 \]
\[ \frac{10 \times 39}{1030} \approx 35.78 \]

\[ w_1 = \frac{x_1 S_1(0)}{v(0)} \]
\[ w_2 = \frac{x_2 S_2(0)}{v(0)} \]

\[ w_1 + w_2 = 1 \]
Short-selling is allowed:

\[ w_1 < 0, \quad w_2 > 100 \%
\]

Take \[ w_1 = 120 \% \quad w_2 = -20 \%
\]

\[ \alpha_1 = w_1 \frac{V(0)}{S_1(0)} = 40 \% \quad \alpha_2 = w_2 \frac{V(0)}{S_2(0)} = -5 \%
\]

\[ V(1) = \alpha_1 S_1(1) + \alpha_2 S_2(2) = 120.5
\]

In reality, security deposit may be required.