A necessary condition for a maximum on the boundary of the region $R \subseteq \mathbb{R}^n$:

A feasible direction $d$ at a point $x \in R$ is a direction such that $x + \varepsilon d$ for all $0 \leq \varepsilon \leq S$, for some $S > 0$.

Necessary condition: a point $x^*$ on the boundary of $R$ is a local maximum of $f$ if $d \cdot \nabla f(x^*) \leq 0$ for all feasible directions $d$.

This test also applies at an interior $x^* \in R$:

$$
\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right) = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \ldots + \frac{\partial f}{\partial x_n} e_n
$$
§ 1.3 Gauss–Jordan Elimination

§ 1.3.1 Pivot Operations

Consider the linear system

\[ \begin{align*}
    x_1 - x_2 - x_3 &= 2 \\
    2x_1 + x_2 - x_3 &= 5 \\
    -x_1 + 2x_2 + 3x_3 &= 0
\end{align*} \]

The solution set of these equations is unchanged by Gauss row operations:

1. multiply any equation/row by a non-zero constant
2. add a (non-zero) multiple of one equation/row to another
3. interchange 2 equations/rows.

We can use these operations to simplify the structure of the equations/matrix.

We won't use operation 3.

We use a combination of 1. & 2.

called a pivot operation or pivot.
Let $a_{ij}$ be any non-zero element of the (augmented) matrix of the system.

We pivot on $a_{ij}$ as follows:

1. Divide row $i$ by $a_{ij}$ — this replaces $a_{ij}$ by 1.

2. Transform the elements $a_{kj}$, $k \neq i$, i.e., the elements in row $k$, $k \neq i$, so 0 by adding multiples of row $i$.

The matrix form of the linear system

$$
\begin{pmatrix}
1 & -1 & -1 & | & -2 \\
2 & 1 & -1 & | & 5 \\
-1 & 2 & 3 & | & 0
\end{pmatrix}
$$

We pivot on element $(2,1)$:

$$
\begin{pmatrix}
1 & 1/2 & -1/2 & | & -9/2 \\
0 & 1/2 & -1/2 & | & 5/2 \\
0 & 2 & 3 & | & 0
\end{pmatrix}
$$

$R_1 := R_1 - R_2/2$

$R_2 := R_2/2$

$R_3 := R_3 + R_2/2$

§ 1.3.2 Gauss-Jordan elimination — see notes.

§ 1.3.3 Systems with more variables than equations.
So consider the system

\[\begin{align*}
  x_1 - x_2 + x_3 - x_4 &= -2 \\
  2x_1 + x_2 - x_3 + x_5 &= 5 \\
  -x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 &= 0
\end{align*}\]

i.e. 5 variables, 3 equations

We are pivoting to express \underline{3 variables} in terms of the other \underline{2}.

As an example, we solve for \((x_1, x_2, x_3)\) as \underline{basic variables} in terms of \((x_4, x_5)\) as \underline{non-basic variables}.

We construct a tableau:

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

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<tr>
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<tbody>
<tr>
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<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>-3</td>
<td>2</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>-2</td>
</tr>
</tbody>
</table>

See notes

\[\begin{align*}
  R_1 &= R_1 \\
  R_2 &= R_2 - 2R_1 \\
  R_3 &= R_3 + R_1
\end{align*}\]
We can go from the tableau back to the equations:

\[ x_1 = 1 + \frac{1}{3} x_4 - \frac{1}{3} x_5 \]
\[ x_2 = 2 - \frac{8}{15} x_4 - 2x_3 x_5 \]
\[ x_3 = -1 + \frac{2}{15} x_4 - \frac{1}{3} x_5 \]

We have expressed the basic variables in terms of the non-basic variables using pivoting.

§1.3.4 Transformation of linear functions

Let \( z = C_1 x_1 + C_2 x_2 + C_3 x_3 + C_4 x_4 + C_5 x_5 + c_0 \)

When we express the basic variables in terms of the non-basic variables we also need to express \( z \) in terms of the non-basic variables.

We do this as follows: let

\[ 2 = x_1 + x_2 - x_5 \]

We rework the famous example with this \( z \). Re-arrange \( z \):

\[ 2 - x_1 - x_2 + x_5 = 0 \]
Our tableau becomes

\[
\begin{array}{ccccccc}
Z & x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
\hline
1 & -1 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 1 & -1 & 0 & 0 \\
0 & 2 & 1 & -1 & 0 & 1 & 1 \\
0 & -1 & 2 & 3 & 1 & 2 & 0 \\
\end{array}
\]

Pursuing to express the basic variables \((x_1, x_2, x_3)\) in terms of the non-basic variables \((x_4, x_5)\) we get

\[
\begin{array}{ccccccc}
Z & x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
\hline
1 & 0 & 0 & 0 & \frac{1}{5} & 2 & 3 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 1 \\
0 & 0 & 1 & 0 & \frac{8}{15} & \frac{2}{3} & 2 \\
0 & 0 & 0 & 1 & -\frac{2}{15} & \frac{1}{3} & -1 \\
\end{array}
\]

Row 1 gives

\[
Z + \frac{1}{5} x_4 + 2x_5 = 3
\]

or

\[
Z = -\frac{1}{5} x_4 - 2x_5 + 3
\]