Chapter 2: Linear Programming

§1.1 A model LP problem

Consider 3 processes to produce 2 drugs. Each process requires the use of a catalyst. Each process produces different amounts of each drug.

<table>
<thead>
<tr>
<th>Process</th>
<th>Amount of catalyst per unit of drug 1, drug 2</th>
<th>Available amount of catalyst</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1, 0</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3, 2</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>0, 2</td>
<td>12</td>
</tr>
</tbody>
</table>

The profits per unit of drug 1 and drug 2 are 3 and 5 respectively.

Let the decision (independent) variables $x_1, x_2$ be the number of units of drug 1 and drug 2 produced. Let $z$ be the profit gained by producing $x_1, x_2$ units of the drugs.

We formulate the problem as follows:

1. **Objective function:** $z = 3x_1 + 5x_2$
2. **Constraints:**
   - Catalyst constraint for process 1: $x_1 + 0x_2 \leq 4$
Thus

\[
\begin{align*}
\text{Max } Z &= 3x_1 + 5x_2 \\
\text{s.t.} \quad &x_1 + 0x_2 \leq 4 \\
&3x_1 + 2x_2 \leq 18 \\
&2x_2 \leq 12 \\
\text{with} \quad &x_1 \geq 0, \ x_2 \geq 0.
\end{align*}
\]

We solve this problem graphically.

§ 1.2 The standard LP problem

\[
\begin{align*}
\text{Max } Z &= c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{s.t.} \quad &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\
\text{with} \quad &x_1 \geq 0, \ x_2 \geq 0, \ldots, \ x_n \geq 0.
\end{align*}
\]

In matrix form we write this as

\[
\begin{align*}
\text{Max } Z &= c^T x \\
\text{s.t.} \quad &Ax \leq b \\
\text{with} \quad &x \geq 0.
\end{align*}
\]
Here

\[
\begin{pmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{pmatrix}, \quad
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}, \quad
\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{pmatrix}
\]

\[
A =
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

We define matrix inequalities elementwise:

\( M \leq N \) if and only if \( M, N \) have the same shape and \( m_{ij} \leq n_{ij} \) for all \( i, j \).

Notes:

1. \( z \) is the objective function.
   \((x_1, \ldots, x_n)\) are the decision variables.
   \((c_1, \ldots, c_n)\) are the cost coefficients.

2. The constraints are linear and all are \( \leq \) (or \( = \) or \( \geq \)).
   \( A \) is the constraint matrix.
   \( b \) is the resource vector.
   Important: \( b \geq 0 \) - non-negativity of \( b \)

3. Non-negativity of the decision variables:
   \( x \geq 0 \).
The set of decision variables \( x \) which satisfy all the constraints form the feasible set. In the standard LP problem, the feasible set is non-empty since \( x = 0 \) is always feasible.

\[ \text{§ 2.2} \quad \text{Graphical solution of the LP problem: } n = 2 \text{ problems} \]

1. Draw the feasible set in the \((x_1, x_2)\)-plane.
2. We use two ways to find the optimal value of the feasible set.

\[ \begin{align*}
  x_1 & \leq 4 \\
  3x_1 + 2x_2 & \leq 18 \\
  2x_2 & \leq 12 \\
  x_1 & \geq 0 \\
  x_2 & \geq 0 \\
  \end{align*} \]

The feasible set is bounded by the constraints. The objective function is \( z = 3x_1 + 5x_2 \leq 15 \) increases.
2 (i) Plot level contours of $Z$.

(ii) List the corner points (vertices) of the feasible region (feasible corner points - FCP's)

<table>
<thead>
<tr>
<th>FCP</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>0</td>
</tr>
<tr>
<td>(4,0)</td>
<td>12</td>
</tr>
<tr>
<td>(4,3)</td>
<td>27</td>
</tr>
<tr>
<td>(2,6)</td>
<td>36</td>
</tr>
<tr>
<td>(0,6)</td>
<td>36</td>
</tr>
</tbody>
</table>

The optimal solution is

$x_1^* = 2, \quad x_2^* = 6, \quad Z^* = 36.$

§ 2.2.2 Observations

1. The feasible set (or region) may be unbounded. Eg

\[-x_1 + x_2 \leq 4\]
\[x_1 - x_2 \leq 2\]
\[x_1 \geq 0, \quad x_2 \geq 0\]

Feasible region unbounded.