Chapter 2

Linear Programming

1 The Standard LP Problem

Linear programming (LP) is the term which is used to describe a wide class of problems in the field of constrained optimization. Broadly speaking, LP seeks to solve the problem of sharing activities amongst limited resources in an optimal way. However, the key feature of LP problems is that all objectives and constraints occur as linear functions of their associated variables.

LP is one of the fundamental problems of Operations Research (OR). Techniques for solving LP problems were developed in military research facilities during World War II. The simplex algorithm, the basic method used for solving LP problems, was developed by George Dantzig in 1947.

1.1 A Model Linear Programming Problem

A pharmaceutical company can produce two types of drug by three different processes. Each process requires the use of a chemical catalyst X which is in very limited supply. Production data are listed in the table below:

<table>
<thead>
<tr>
<th>Process</th>
<th>Drug 1</th>
<th>Drug 2</th>
<th>Availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>Profit</td>
<td>3</td>
<td>5</td>
<td>× $1000</td>
</tr>
</tbody>
</table>

Thus process 1 produces only drug 1, uses 1 gm of X per unit produced and has only 4 gm of X available. Similar interpretations apply for processes 2 and 3. Drug 1 sells at a unit profit of $3000 while drug 2 sells for a unit profit of $5000. The problem is to find the number of units of each drug to produce in order to maximize the total profit.

Mathematical Formulation: Let $x_1, x_2$ be the number of units of each drug which are to be produced, and let $Z$ be the total profit in units of $1000. Then the LP formulation of the drug problem is as follows:
All points of the line segment $C_1C_2$ are optimal with the optimal objective function value $Z^* = 36$. Thus, if there is more than one optimal solution there will be an infinite number of them. Geometrically, this condition can occur when the level contours of the objective function are parallel to a boundary line of the feasible region.

To represent the solution mathematically let $x_1 = t \geq 0$. The boundary line is given by equality in the second constraint $3x_1 + 2x_2 \leq 18$, which implies $x_2 = 9 - \frac{3}{2}x_1 = 9 - \frac{3}{2}t$. Since $x_2 \geq 0$ we must have $t \leq 6$. The first constraint, $x_1 \leq 4$, implies $t \leq 4$ and the third constraint, $2x_2 \leq 12$, implies $x_2 \leq 6$ or $t \geq 2$. Thus $x_1^* = t$, $x_2^* = 9 - \frac{3}{2}t$, where $2 \leq t \leq 4$.

Since an optimal solution is a corner point of the feasible region, it is clear that an algorithm to solve the standard LP problem need only consider feasible corner points (FCP's). Observe that
II. *Iteration Steps*: Move to an adjacent FCP with the best potential $Z$ increase.

III. *Stopping Rule*: Stop at FCP* if its $Z^*$ is greater than the $Z$ values of all its adjacent FCP’s.

For the drug problem the simplex algorithm might proceed as follows:

1. Start at FCP $F_1(0, 0)$ with $Z = 0$. See Figure 2.5.

2. Move to $F_3(0, 6)$ since $Z$ increases at a faster rate along $x_1 = 0$ than along $x_2 = 0$ [towards $F_2(4, 0)$]. Move to $F_4(2, 6)$ with $Z = 36$ for the same reason ($Z$ actually decreases going back to $F_1$).

3. Stop at $F_4^*$ since $Z_5 < Z_4^* > Z_3$. ($Z_5 = 30, Z_3 = 27$).

We now investigate the algebra of the simplex algorithm. In particular, how do we characterize FCP’s algebraically?