Tutorial 10 — Week 11

Tutorial questions are starred. Advanced questions are marked A.

1. Let \( u(\tilde{x}) = x'\tilde{a} \) and \( v(\tilde{x}) = \frac{1}{2} x' A \tilde{x} \) where \( \tilde{a} \) is a constant \( n \times 1 \) vector and \( A \) is a constant symmetric \( n \times n \) matrix. Prove that

\[
\frac{\partial u}{\partial \tilde{x}} = \tilde{a} \quad \text{and} \quad \frac{\partial v}{\partial \tilde{x}} = A \tilde{x}.
\]

2. Let \( \tilde{x}_1 \) and \( \tilde{x}_2 \) be the allocation vectors of any two feasible \( n \)-asset portfolios. Show that the covariance of their random returns \( R_1 \) and \( R_2 \) is given by

\[
cov\{R_1, R_2\} = \tilde{x}_1' S \tilde{x}_2.
\]

3. Let \( IC(t) \) denote the indifference curve for an investor with risk-aversion parameter \( t \). Prove that the efficient frontier for unrestricted portfolios given parametrically by \( \mu = \mu(t); \sigma = \sigma(t) \) is everywhere tangential to \( IC(t) \).

4. Obtain the efficient frontier for the 3-asset portfolio with parameters:

\[
\tilde{r} = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.8 \end{pmatrix}; \quad S = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}
\]

Derive the corresponding optimal allocation vector \( \tilde{x} \) as a function of the risk aversion parameter \( t \). Find the allocation to the minimum risk portfolio. Sketch the critical line in the \( x_1 x_2 \)-plane.

5. Use the result of Q2. above to prove that if \( P_1 \) is a feasible unrestricted portfolio with return \( R_1 \) and \( P_2 \) is an efficient unrestricted portfolio with return \( R_2 \), then (in the standard notation),

\[
cov\{R_1, R_2\} = \frac{\tilde{a}}{2}(r_1 - \frac{b}{\tilde{a}})(r_2 - \frac{b}{\tilde{a}}) + \frac{1}{\tilde{a}}.
\]

6. Let \( P \) be an efficient unrestricted portfolio. Portfolio \( Q \) is obtained from \( P \) by extending the tangent to the efficient frontier at \( P \) in the \( \mu \sigma \)-plane to the point \( P_0 \) on the \( \mu \)-axis and then finding \( Q \) on the MVF such that \( Q \) has the same expected return as \( P_0 \). Prove, using the result of Question 5 above, that

\[
cov\{R_P, R_Q\} = 0.
\]
7. *(a) For a certain 3-asset unrestricted portfolio, investor-\(A\) with \(t_A = 1\) selects an optimal portfolio \(\bar{x}_A = (\frac{1}{2}, \frac{1}{2}, 0)\); investor-\(B\) with \(t_B = 2\) selects an optimal portfolio \(\bar{x}_B = (\frac{1}{2}, 0, \frac{1}{2})\). Find the allocation vector \(\bar{x}_C\) for Ms Chicken with \(t_C = 0\).

(b) Which investor will select an efficient portfolio with \(\bar{x} = (\frac{1}{2}, -\frac{1}{2}, 1)\)?

(c) Show that there is no efficient portfolio with \(\bar{x} = (\frac{1}{3}, \frac{1}{3}, 1)\).

8. *A An investor seeks an efficient \(n\)-asset portfolio with a fixed return equal to \(\rho\). Otherwise the only other constraint is the budget condition. Show by solving a QP-problem with two equality constraints, that (in the usual notation) the optimal allocation vector is the solution of:

\[ S\bar{x} = \left( \frac{c-b\rho}{d} \right) \bar{c} + \left( \frac{a\rho-b}{d} \right) \bar{r} . \]

Show that the same result is obtained from the optimal portfolio equations derived in lectures in terms of the investor’s risk aversion parameter.