Tutorial 9 — Week 10

Tutorial questions are starred. Advanced questions are marked A.

1.∗ Consider a 2-asset portfolio with parameters:
\[ r_1 = 10, \quad r_2 = 20, \quad s_1 = 2, \quad s_2 = 6, \quad \rho = 1. \]

Show that the feasible set consists of the pair of straight lines
\[ \mu = 5(1 \pm \frac{1}{2} \sigma). \]

(a) Obtain (i) the unrestricted efficient frontier (EF). Find the restricted EF’s when
(ii), \( x_1 \geq 0 \); when (iii), \( x_2 \geq 0 \) and when (iv), both \( x_1 \geq 0 \) and \( x_2 \geq 0 \).

(b) An investor has risk-aversion parameter \( t = 3.2 \). Find the investor’s optimal portfolio in each of the four cases above.

2.∗ Prove that for a general 2-asset portfolio \((r_1, s_1), (r_2, s_2)\):
\[ \sigma_{\text{min}} = s_1 s_2 \sqrt{1 - \rho^2 \over s_1^2 + s_2^2 - 2 \rho s_1 s_2}. \]

Show also that \( \sigma_{\text{min}} \leq \min(s_1, s_2) \) and that \( \sigma_{\text{min}} = 0 \) if and only if the correlation coefficient \( \rho = \pm 1 \).

3. Obtain the feasible set, minimum variance frontier, efficient frontier and minimum risk portfolio for each of the following two asset portfolios:

(a) \( r_1 = r_2 \) and \( s_1 < s_2 \).

(b) \( s_1 = s_2 \) and \( r_1 < r_2 \) and \( \rho = \pm 1 \).

4. Given two risky assets \( P_1 \) and \( P_2 \) with \( r_1 > r_2 \), \( s_1 > s_2 \) and correlation coefficient \( \rho \) show that an investor with risk aversion parameter \( t \) will allocate a proportion
\[ x_1 = \frac{(r_1 - r_2)t + s_2^2 - \rho s_1 s_2}{s_1^2 + s_2^2 - 2 \rho s_1 s_2} \]
of wealth to asset \( P_1 \) and a proportion
\[ x_2 = \frac{(r_2 - r_1)t + s_1^2 - \rho s_1 s_2}{s_1^2 + s_2^2 - 2 \rho s_1 s_2} \]
to asset $P_2$.

Determine the value of $t$ for an investor indifferent to the two risky assets and find the corresponding proportions $x_1$ and $x_2$ if this investor chooses to invest in both risky assets.

5.* Use the general $n$-asset formula $\sigma^2 - \frac{a}{d}(\mu - \frac{b}{a})^2 = \frac{1}{a}$ to obtain the unrestricted minimum variance frontier for the 2-asset portfolio with parameters:

$$r_i = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \quad S_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Sketch the feasible set in the $\mu\sigma-$plane and indicate on your diagram the position of the two assets. Find the parameters of the minimum risk portfolio $P_0$.

Obtain the corresponding parametric equations of the feasible set in the form $\mu = \mu(t)$ and $\sigma = \sigma(t)$ in terms of the risk aversion parameter $t$. What values of $t$ identify the assets $P_1$ and $P_2$ and the MRP $P_0$?

6. Prove that if $x_1, x_2$ with $x_1 + x_2 = 1$ give the proportion of wealth invested in two assets that deliver a percentage return $R_1$ and $R_2$ respectively, then the portfolio return is given by

$$R = x_1 R_1 + x_2 R_2.$$

Generalise this result to the case of $n$ assets.