1 [15 marks]
(a) A company makes two types of bookshelf, a deluxe one out of craft wood and a budget one out of particle board. It can sell all it produces. The deluxe bookshelf sells for $60 and the budget for $40. The materials and overheads in a deluxe bookshelf cost $14 and in a budget bookshelf $10. The deluxe bookshelf takes 0.4 hours to machine and 0.2 hours to assemble; the budget bookshelf takes 0.2 hours to machine and 0.2 hours to assemble. Machining costs $30 per hour and assembling costs $20 per hour. If the company has 24 hours of machining per week and 16 hours of assembling per week, how many bookshelves of each type should it produce to maximise its profit?

Formulate this problem mathematically and then solve it graphically.

(b) Consider the following linear programming problem,
\[
\text{Max } Z = 2x_1 + x_2 + 2x_3
\]
subject to
\[
2x_1 + 2x_2 + x_3 \leq 18
\]
\[
-x_1 - 4x_2 - 2x_3 \geq -8
\]
with \( x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \).

(i) Put this problem into the standard form and then solve it using the simplex algorithm.

(ii) Write down the dual form of the problem (but do not solve it).

(iii) From the optimal tableau of the primal problem in Part (i) write down the optimal solution of the dual problem.

(c) The following tableau was obtained after several steps of the simplex algorithm in a standard linear programming problem. Attempt to complete the algorithm and interpret your results fully.

<table>
<thead>
<tr>
<th>Z</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>7</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

(d) By introducing appropriate variables put the following problem into form suitable for the two-phase simplex algorithm. Hence write down an initial feasible corner point for phase-1. Do not actually implement the algorithm.
\[
\text{Min } Z = -2x_1 - 3x_2 - x_3
\]
subject to
\[
2x_1 - 5x_2 + 2x_3 = 24
\]
\[
x_1 + x_2 - 3x_3 \leq -7
\]
with \( x_1 \geq 0, x_3 \geq 0 \).
2 [15 marks]
(a) Question on nonlinear optimisation.
(b) A factory manufactures two types of chair $X$ and $Y$ from metal tubing. The factory can purchase up to 52 units of tubing at a cost of $1 per unit. Six units of the tube can be processed into one unit of $X$ at a processing cost of $4 for each unit of $X$ produced, and ten units of tube can be processed into one unit of $Y$ at a processing cost of $8 for each unit of $Y$ produced. If $x$ units of $X$ are produced and $y$ units of $Y$ are produced, $X$ sells for $(34 - x - 2y)$ per unit and $Y$ sells for $(58 - 2y)$ per unit. The factory wishes to maximise its profit.

(i) Find the cost in dollars of tubing and the cost in dollars of processing to produce $x$ units of $X$ and $y$ units of $Y$. Hence show that the profit in dollars is given by

$$f(x, y) = x(24 - x - 2y) + y(40 - 2y).$$

Formulate the problem as a nonlinear programming problem subject to constraints.

(ii) Convert the problem in Part (i) to a minimisation problem and show that the corresponding Lagrangian function, ignoring the non-negativity conditions, is

$$L = x^2 + 2xy + 2y^2 - 24x - 40y + \lambda(6x + 10y + s^2 - 52)$$

where $\lambda \geq 0$.

(iii) Use the Lagrangian function to determine necessary conditions for maximum profit. Hence show that the factory maximises its profit if it produces 2 units of $X$. How many units of $Y$ should the factory produce to maximise its profit?

(iv) Why are the conditions in Part (iii) also sufficient?

3 [15 marks]
(a) A 2-asset portfolio has parameters

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ 1 - x \end{pmatrix}.$$

(i) Use the portfolio equations to find $\mu(x)$ and $\sigma(x)$, and hence show that the feasible set consists of the pair of straight lines $\mu = \frac{1}{4}(5 \pm \sigma)$, $\sigma \geq 0$, in the $\mu\sigma$-plane.

(ii) Sketch the feasible set, the location of the two assets $A_1$ and $A_2$ and find the minimum risk portfolio $P_0$ in the $\mu\sigma$-plane. What is the allocation to $A_1$ and $A_2$ for $P_0$? Carefully indicate on your sketch which investors allow short-selling of $A_1$, $A_2$ and no short-selling of either.

(iii) ^A What is the optimal portfolio for an investor who has risk-aversion parameter $t = 8$?
(b) A 3-asset Markowitz portfolio has parameters
\[
\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]

Obtain the following:

(i) The minimum variance and efficient frontiers in the \( \mu \sigma \)-plane.

(ii) The allocation vector \( \mathbf{x} \) as a function of the risk-aversion parameter \( t \).

(iii) The projection of the critical line onto the \( x_1x_3 \)-plane.

(iv) The parameters for the minimum risk portfolio.

(v) The investors who short-sell the first asset.

(c) Question on restricted portfolios. A sample advanced question:

Write down the Lagrangian for the 3-asset portfolio in Part (b) if there is no short-selling of asset 1. Derive the first-order Kuhn-Tucker conditions for the optimal portfolio. Hence show that the \( x_i \) can be written as
\[
x_1 = -t + \frac{1}{2} + \frac{1}{2} \ell_1
\]
\[
x_2 = \frac{1}{4} - \frac{1}{4} \ell_1
\]
\[
x_3 = t + \frac{1}{4} - \frac{1}{4} \ell_1.
\]

Determine \( \ell_1 \) and hence derive and sketch the critical line in the \( x_1x_3 \)-plane.

4 [15 marks]

(a) A portfolio of risky assets has parameters \( a = 1, b = 3, c = 10 \). Find the feasible set in the \( \mu \sigma \)-plane.

(i) A riskless asset with rate of return \( r_0 = 2 \) units for both lending and borrowing is added to the portfolio. Find the capital market line.

(ii) State the one fund theorem in terms of the market portfolio \( P_M \). Prove that \( t_M = 1/(b - a r_0) \). Use this result to find \( \mu_M \) and \( \sigma_M \).

(iii) Determine the allocation to the riskless and risky assets of an investor with a risk aversion parameter \( t = 1/5 \).

(b) Question on dynamic programming.

END OF EXTENDED ANSWER QUESTIONS.