In the case of fractals and similarly complex objects, quantities such as length, area and volume may be infinite, or rather undefined. Hence it is necessary to consider other notions of measurement when comparing fractals. This report aims to motivate such a measure and show that it contains many of the intuitive properties of dimension. Furthermore, some methods for measuring this 'fractal dimension' are described.

1. Introduction

Length, area and dimension have classically been fairly straightforward to compute for an object in two or even three dimensions as it is always possible to approach a true value by measuring with increasing precision. However, this is not the case for fractals as these quantities may be undefined. These ideas can be seen through the more familiar curves of coastlines and borders which are 'statistically self-similar'. That is, a portion of the curve can, in a statistical sense, be considered a scaled down version of the whole [4]. Furthermore they can be complex enough to be treated as fractals. As demonstration of this perhaps unintuitive idea: Lewis Richardson, while attempting to find a relationship between the probability of two countries going to war and the length of their shared border, found significant discrepancies in quoted lengths. Using the border between Spain and Portugal as an example, a Spanish encyclopedia gave 990 km while a Portuguese encyclopedia gave 1220 km [5]. This led him to further investigate this, with findings that will be explored in this report.

2. Measuring fractal curves and power laws

The complexity and self-similarity properties of coastlines were demonstrated in Richardson’s report [6]. These were explored through measurements of coastlines.

2.1. The coastline of Britain. Consider measuring the coastline of Britain. In order to do this practically, a geographer may take a pair of compasses and walk it around the curve as a measuring stick. This gives a polynomial approximation of the curve. The compass length can then be changed to give a range of measurements.
Doing this for the coastline of Britain gives the following table

<table>
<thead>
<tr>
<th>Compass Setting (km)</th>
<th>Length (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>2600</td>
</tr>
<tr>
<td>100</td>
<td>3800</td>
</tr>
<tr>
<td>54</td>
<td>5770</td>
</tr>
<tr>
<td>17</td>
<td>8640</td>
</tr>
</tbody>
</table>

Table 1. Approximations for the coastline of Britain [5]

Hence as the compass setting becomes smaller and so the ‘precision’ in some sense improves, the total measured length increases dramatically. This significant change in length, depending on the scale at which it is measured, suggests that length may be a poor measurement for comparing curves such as these.

Now this can be done similarly for a circle.

<table>
<thead>
<tr>
<th>Compass Setting (km)</th>
<th>Length (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>2600</td>
</tr>
<tr>
<td>259</td>
<td>3106</td>
</tr>
<tr>
<td>131</td>
<td>3133</td>
</tr>
<tr>
<td>65</td>
<td>3139</td>
</tr>
<tr>
<td>32</td>
<td>3141</td>
</tr>
<tr>
<td>16</td>
<td>3141</td>
</tr>
</tbody>
</table>

Table 2. Measurements for the circle [5]
Here there is very little change in the total length and in fact the total length appears to converge. Comparing these on a log-log graph of length against compass setting along with other coastlines:

As seen in Fig. 2, these points are fairly well-fitted by straight lines. This gives a horizontal line for the circle and a line with gradient \( d \approx -0.36 \) for the coast of Britain. Let \( u \) denote the total length and \( s \) the compass setting. Then extrapolating this gives the relation

\[
\log(u) = d \log(s) + b \tag{2.1}
\]

Rearranging gives the power law

\[
u = c \cdot s^d \tag{2.2}
\]

for some constant \( c \). Then for the coastline of Britain

\[
u \propto \frac{1}{s^{0.36}} \tag{2.3}
\]

From this relation we immediately see that as the compass setting \( s \to 0 \), \( u \to \infty \). Hence the true length is in some sense infinite.

Of course for an actual map the precision will always be limited by its resolution. Furthermore, for small enough compass settings the coastline will become undefined with difficulties such as tide, river deltas and start/end points. However, for reasonable compass settings this characteristic power law gives a measure of how the total length increases.
as finer details are taken into account. In this sense, the power law is a measure of the complexity of the curve. In particular, you would expect that a more complex curve would have greater detail at smaller scales, and so the total length would increase more quickly. In this case, a lower \( d \) corresponds to greater complexity.

Now take \( D = 1 - d \) as the measure of complexity. Then, the ‘complexity’ of the circle is 1 and for the coastlines it is between 1 and 2. With this measure, the complexity increases with greater fine detail. We can now see some relationships to dimension and so refer to this as a measure of ‘fractal dimension’, namely \textit{compass dimension}. Using this gives a compass dimension of 1.36 for the coastline of Britain. Further relationships and a justification for this choice will be considered later (see section 4: Self-similarity dimension).

2.2. Extending to fractals. Now extending this idea to fractals, consider the Koch curve\(^1\). Take compass settings of length \( s = \frac{1}{3} \). Each time the compass setting is decreased by a factor of 1/3, the sections which were previously considered straight lines will be taken into account as being comprised of 4 lines of length \( \frac{1}{3} \), as in the construction. This will give total lengths of \( \frac{4}{3} \). Taking the logarithm of both sides

\[
D = \frac{\log(4)}{\log(3)} \approx 1.26
\]  
(2.4)

3. Dimension

There are many notions of dimension including topological, Hausdorff, self-similarity, box-counting, capacity, information and Euclidean, among others. Some will be more useful in some circumstances than others, though even when two definitions may be applicable they will not necessarily agree \cite{5}. Hence it is clear that it is difficult to construct an overarching framework for dimension that will be universally correct. However, the most important property of any definition of dimension is that it agrees with intuitive notions of dimension, i.e. Euclidean \( \mathbb{R}^n \).

3.1. Fractal dimension. As seen, whilst measures such as length may be infinite or undefined for fractals, it is possible to define a measurement for the degree of complexity, given by \( D \). \textit{Compass dimension} has previously been demonstrated, next a few more methods for measuring fractal dimension will be explored.

4. Self-similarity dimension

The self-similarity dimension uses the self-similar property of fractals. It is generally applicable when a fractal is constructed by a scaling and translation in space, such as the Koch curve and Cantor set.

\(^1\)See the appendix for details on the Koch curve
4.1. **Method.** To determine the self-similarity dimension, consider scaling an object by a contraction factor $s$, then attempting to fit this scaled object into the original object $a$ times. Then

$$a = \frac{1}{s^D}$$  

(4.1)

The fractal dimension will then be given by

$$D = \frac{\log(a)}{\log(1/s)}$$  

(4.2)

4.2. **Line, square and cube.** To demonstrate that this can be used as a valid notion of dimension, consider scaling a line, square and cube by reduction factors of $1/3$, $1/6$ and $1/n$.

<table>
<thead>
<tr>
<th>Object</th>
<th>Scaling factor</th>
<th>Number of pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line</td>
<td>$1/3$</td>
<td>3</td>
</tr>
<tr>
<td>Line</td>
<td>$1/6$</td>
<td>6</td>
</tr>
<tr>
<td>Line</td>
<td>$1/n$</td>
<td>$n$</td>
</tr>
<tr>
<td>Square</td>
<td>$1/3$</td>
<td>9</td>
</tr>
<tr>
<td>Square</td>
<td>$1/6$</td>
<td>36</td>
</tr>
<tr>
<td>Square</td>
<td>$1/n$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Cube</td>
<td>$1/3$</td>
<td>27</td>
</tr>
<tr>
<td>Cube</td>
<td>$1/6$</td>
<td>216</td>
</tr>
<tr>
<td>Cube</td>
<td>$1/n$</td>
<td>$n^3$</td>
</tr>
</tbody>
</table>

**Table 3.** Scaling n-cubes

From this, there is a clear power law

$$a = \frac{1}{s^D}$$  

(4.3)

with $D = 1$, 2, 3 for the line, square and cube. If we consider reducing n-dimensional Euclidean space by a factor $k$, then in each linearly independent direction an object in this space will be contracted by the factor $k$. Then in each direction, the object could ‘fit’ into itself $1/k$ times. This occurs $n$ times, giving $a = \frac{1}{k^n}$. Furthermore, consider an n-dimensional manifold in some higher dimensional space. This will be locally Euclidean of n-dimension, so contracting in each possible direction will be equivalent to contracting in n-dimensions. Hence, this scaling definition of dimension gives the expected values for objects in $\mathbb{R}^n$. 


4.3. Koch curve. Now consider doing the same for the Koch curve. Consider scaling by a factor of $\frac{1}{3}$. By the construction of the Koch curve, this will fit in 4 times. Hence

$$a = \frac{1}{s^D}$$

(4.4)

$$4 = \frac{1}{3^{-D}}$$

(4.5)

$$D = \frac{\log(4)}{\log(3)}$$

(4.6)

$$\approx 1.26$$

(4.7)

Hence we see that this agrees with the previously found fractal dimension.

In fact, this can be used to justify the compass dimension, in particular the choice of $D = 1 - d$.

Let the total length be denoted $u$, the scaling factor $s$ and number of pieces be $a$. Recall the compass power law ($u = c \cdot s^d$) and that of the self-similarity dimension ($a = \frac{1}{s^D}$). Choose appropriate units such that $c = 1$ (i.e. the total length of the curve is 1) so that $s$ coincides with the contraction factor. Then the total measured length is equal to the number pieces times the scaling factor. That is, $u = a \cdot s$. Then

$$\log(u) = \log(a) + \log(s)$$

(4.8)

$$\log(u) = d \log(s)$$

(4.9)

$$\log(a) = -D \log(s)$$

(4.10)

$$d \log(s) = -D \log(s) + \log(s)$$

(4.11)

$$D = 1 - d$$

(4.12)

5. Box-counting Dimension

Now consider a ‘wild fractal’ in the plane which may have scaling properties, such as Fig. 3 (left). For this, it is not possible to consider compass settings as there is no single curve. Nor is there sufficient self-similarity to use the self-similarity dimension. Instead, it is necessary to use the box-counting method. Because this is easily applicable in such general cases without the need for self-similarity or a single curve, this method is very useful in practice.

5.1. Method. Put the object onto a grid with perpendicular lines of distances $\delta$. Then count the number of grid boxes that contain part of the fractal, denoted $N_\delta$. Plot $\log(N_\delta)$ against $\log(1/\delta)$. Attempt to fit a straight line to the data. Then the slope $D$ gives the dimension. That is, for some constant $b$

$$\log(N_\delta) = D \log(1/\delta) + b$$

(5.1)
Doing this for the coastline of Britain using two grid settings gives $D \approx 1.31$, which is in reasonable agreement with 1.36 found by the compass dimension.

\[ d(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2} \]  

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6. Hausdorff Dimension

The Hausdorff dimension gives an overarching definition for fractal dimension and can in fact be used as a measure for metric spaces. However, for simplicity consider a set $A$ in Euclidean space $\mathbb{R}^n$. This notion of dimension, whilst important from a theoretical standpoint, is very difficult to use in practice.

6.1. Definition. First, note the following definitions:

The distance between any two points in $A$

\[ d(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2} \]  

Next is the diameter of a subset $U$ in $\mathbb{R}^n$

\[ \text{diam} U = \sup \{ d(x, y) \mid x, y \in U \} \]  

This is essentially the maximum distance between any two points in $U$. An open cover of $A$ is a countable family of open subsets of $U$ ($U_1, U_2, ...$) such that $A$ is a subset of the union of $U_i$.

Consider

\[ h^s_e(A) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(U_i)^s \mid \{U_1, U_2, ...\} \text{ form an open covering of } A \text{ with } \text{diam}U_i < \epsilon \right\} \]
Taking $\epsilon \to 0$ gives the $s$-dimensional Hausdorff measure of $A$

$$h^s(A) = \lim_{\epsilon \to 0} h^s_\epsilon(A)$$  \hspace{1cm} (6.4)

This essentially involves finding a covering of the set which minimises the sum of $\text{diam}(U_i)^s$. For $\mathbb{R}^n$, this is equivalent to covering with $n$-dimensional balls as these will give the maximum volume for a given diameter. Hausdorff proved that for $t > s$, if $h^s(A) < \infty$ then $h^t(A) = 0$.

Furthermore there exists a point $s$ such that

$$h^s(A) = \begin{cases} 
\infty & \text{if } s < D_H \\
0 & \text{if } s > D_H 
\end{cases}$$  \hspace{1cm} (6.5)

where $D_H$ is defined to be the Hausdorff dimension.

Hence the graph of $h^s(A)$ appears as:

![Graph of Hausdorff measure](image)

**Figure 4.** Hausdorff dimension plot [3]

Then,

$$D_H = \sup \{ s \mid h^s(A) = \infty \}$$  \hspace{1cm} (6.6)

Note that $h^{D_H}(A)$ takes non-negative real numbers, including 0 and $\infty$.

6.2. **Curve.** In order to develop a more intuitive understanding of Hausdorff dimension, consider a curve in $\mathbb{R}^n$ of length $L$. Now aim to cover it with balls such that the sum of the diameters to the power of $s$ is minimised. This will be when the sets perfectly cover the line.
Assume that this is possible with all diameters equal, given by $\delta$. The number of coverings of diameter $\delta$ will be $L/\delta$. Then considering $\sum_{i=1}^{\infty} \delta_i^s$ with $\delta < \epsilon$,  

$$h^s(A) = \frac{L}{\delta} \delta^s$$

$$= L\delta^{s-1}$$

$$h^s(A) = \lim_{\delta \to 0} L\delta^{s-1}$$

Hence

$$h^s(A) = \begin{cases} 
\infty & \text{if } s < 1 \\
L & \text{if } s = 1 \\
0 & \text{if } s > 1 
\end{cases}$$

Thus $D_H = 1$ as expected for a curve. Note that it is generally not appropriate to assume that all coverings will be of equal diameter. Lifting this assumption will be demonstrated in the following example.

6.3. **Square.** From Hausdorff’s result, if $h^s(A) \neq 0$ or $\infty$, then $s = D_H$. Consider a perfect covering of a square of side lengths $L$. Assume this is possible with countably many circles.
Then the sum of the diameters of the circles must be equal to $L^2$. i.e.

$$\sum_{i=1}^{\infty} \frac{\pi \delta_i^2}{4} = L^2 \quad (6.11)$$

$$\sum_{i=1}^{\infty} \delta_i^2 = \frac{4}{\pi} L^2 \quad (6.12)$$

Take $s = 2$ as this is the expected dimension.

$$h^2(A) = \lim_{\epsilon \to 0} \frac{4}{\pi} \frac{L^2}{\epsilon^2} \quad (6.13)$$

$$= \frac{4}{\pi} L^2 \quad (6.14)$$

This is not equal to 0 or $\infty$, so $D_H = 2$

6.4. As a measure. Note that the Hausdorff dimension of the curve gave the length, whilst that for the square gave $\frac{4}{\pi}$ times the area. In fact, by considering an n-dimensional object $A_n$ in Euclidean space covered by n-dimensional balls

$$c_n h^n(A_n) = vol^n(A_n) \quad (6.15)$$

where $c_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}$ is the volume of an n-dimensional ball of diameter 1 \[^3\] and $vol^n(A_n)$ is the Lebesgue measure, i.e. the normal n-dimensional volume.

\[^2\]Here $\Gamma(x)$ is the Gamma function
6.5. **Relationships to other fractal dimensions.** The Hausdorff dimension is strongly related to the Self-similarity dimension in the following way.

If $F \subset \mathbb{R}^n$ and $\lambda > 0$ then

$$h^s(\lambda F) = \lambda^s h^s(F)$$  \hspace{1cm} (6.16)

where $\lambda F = \{\lambda x : x \in F\}$. That is, by scaling the fractal by a factor $\lambda$, the $s$-dimensional Hausdorff measure scales by $\lambda^s$. Taking $s = D$ where $D$ is the Hausdorff dimension,

$$\frac{h^D(F)}{h^D(\lambda F)} = \frac{1}{\lambda^D}$$  \hspace{1cm} (6.17)

Considering $\lambda$ as a contraction factor, the $D$-dimensional Hausdorff dimension will give the $D$-dimensional volume of the set. That is, $\frac{h^D(F)}{h^D(\lambda F)} = a$ where $a$ is the number of copies of the scaled set that ‘fit’ into the original object. Hence $a = \frac{1}{\lambda^D}$ as for the self-similarity dimension.

The Hausdorff dimension can also be used to justify the box-counting dimension to some extent, though they do not agree for all sets. Box-counting attempts to avoid the difficulty of determining $\text{diam}(U_i)^s$ by instead replacing it by $\delta^s$. That is, it assumes that the minimal covering of the set will be given by squares of side length $\delta$.

Recall that the box-counting dimension $D_b$ is given by

$$\log(N_\delta) = D_b \log(1/\delta) + b$$

where $N_\delta$ is the number of boxes needed to cover the set and $\delta$ is the side length of these boxes as $\delta$ and hence the diameter of these boxes approaches 0. As $\delta \to 0$, both sides will tend to infinity and $b$ will no longer influence the relation. Then for $\delta \to 0$

$$\log(1/\delta)D_b = \log(N_\delta(A))$$  \hspace{1cm} (6.18)

$$\frac{1}{\delta^{D_b}} = N_\delta(A)$$  \hspace{1cm} (6.19)

$$N_\delta(A)\delta^{D_b} = 1$$  \hspace{1cm} (6.20)

$$\sum \delta^{D_b} = 1$$  \hspace{1cm} (6.21)

That is, summing over the number of ‘boxes’. Then $\sum_{i=1}^{\infty} \text{diam}(U_i)^s$ can be computed with $\text{diam}(U_i) = \sqrt{2}\delta$.

7. **Conclusion**

Fractals are objects with such complexity that usual means of measurement cannot be applied. Instead, considering a measure of complexity is more appropriate. In fact, this can be defined such that it obtains many of the properties of ‘dimension’. This report aimed
to demonstrate this through the fractal nature of coastlines, then develop some definitions and methods for measuring this ‘fractal dimension’. In particular, compass dimension, self-similarity dimension, box-counting dimension and Hausdorff dimension.
8. Appendix

8.1. **Koch curve.** The Koch curve is constructed by starting with a straight line. At each step, removing the middle third and replacing it with the upper section of the equilateral triangle with the same base length as the removed section as shown. This is repeated *ad infinitum.*

![Koch curve construction](Copyright 2013 Fractal Foundation)
References


