Pascal’s Triangle and Divisibility

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Outline

- Recall Pascal’s Triangle?
- Cellular Automata
  - One Dimensional
  - Two Dimensional
- Divisibility
  - Kummer’s Result
  - Lucas’ Criterion
  - Legendre’s Identity
- Fundamental Theorem of Pascal’s Triangle
What is it?

Source: http://www.conservapedia.com/Pascal’s_triangle
Binomial Coefficients

- Coefficients from expansion of \((1+x)^n\)
- Also the digits in Pascal’s Triangle
- Can be derived:
  - Recursively
  - Directly

\[
\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} x^k
\]

Source: http://www.mathsisfun.com/pascals-triangle.html
Cellular Automata

- Perfect Feedback Machines
- “Mathematical finite state machines which change the state of their cells step by step”
  - From “Chaos and Fractals: New Frontiers of Science”
- Each cell has ‘p’ possible states
  - Mostly 2-state automata

1 2
1-D

- Symbolised by a row of cells

- Needs:
  - Initial State
  - Formula

- Partial independence
  - Similar to chaos game

Source: Chaos and Fractals: New Frontiers of Science
*not* the Sierpinski Triangle
2-D

- Symbolised by a grid
- Needs:
  - Initial State
  - Formula
- Lots of examples
  - 1-in-8 rule

Source: Chaos and Fractals: New Frontiers of Science
Conway’s Game of Life

- John Horton Conway
- B3/S23
- Patterns:
  - “Stills”
  - “Blinkers”
  - “Gliders”

Source: http://en.wikipedia.org/wiki/Conway%27s_Game_of_Life
Conway’s Game of Life

- This was worth $5,000

http://upload.wikimedia.org/wikipedia/commons/e/e5/Gospers_glider-gun.gif
Conway’s Game of Life

Fewest cells

5x5 square

One cell high
p-state Automata

- They exist!!
- Often simplified to 2-state
  - One state one colour, other states a different colour
- Helpful for building fractals
http://mathlesstraveled.com/2012/10/20/visualizing-pascals-triangle-remainders/
Divisibility

- Kummer’s Result
- Lucas’ Criterion
- Legendre’s Identity
- All aid in constructing fractals by divisibility
\[(n, k) \neq \binom{n}{k} \quad \implies \quad (n, k) = \binom{n+k}{k} \]

\[
\binom{n+k}{k} = \frac{(n+k)!}{n!k!}
\]
Kummer’s Result

- Carry Function

\[ c_p(n, k) \]

- The Result:

\[ r = c_p(n, k) : \binom{n + k}{k} \text{ is divisible by } p^r \text{ but not } p^{r+1} \]
Kummer’s Result

\[ n = 8, \quad k = 7 \]

\[ p = 33 \]

\[ \begin{bmatrix}
1 & 0 & 1 & 0 & 3 \\
0 & 1 & k & 1 & 3
\end{bmatrix} \]

\[ \frac{1}{2} \left( \begin{array}{c}
1 \\
1 \\
3 \\
0
\end{array} \right) = \frac{1}{14} \left( \begin{array}{c}
17 \\
17 \\
14 \\
0
\end{array} \right) = \frac{2}{2} \left( \begin{array}{c}
2 \\
2 \\
5 \\
0
\end{array} \right) = \frac{1}{12} \left( \begin{array}{c}
1 \\
1 \\
11 \\
12
\end{array} \right) = \frac{1}{13}
\]

\[ c_2(8, 7) = 0 \]
\[ c_3(8, 7) = 2 \]
\[ c_5(8, 7) = 1 \]
\[ c_{11}(8, 7) = 1 \]
\[ c_{13}(8, 7) = 1 \]
Lucas’ Criterion

- More useful for old coordinate system
- Can still be applied for the new system

\[ n = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_0 \]
\[ (a_m a_{m-1} \cdots a_0)_{2} \]
\[ k \]
\[ b_m b_{m-1} \cdots b_0 \]
\[ (b_m b_{m-1} \cdots b_0)_{2} \]
\[ a_i \geq b_i \quad \left( \binom{n}{k} k \right) \text{ is odd!} \]
\[ \binom{n}{k} \]
Lucas’ Criterion

\[ n = 15, \quad k = 7 \]

\[ \binom{n}{k} = \binom{15}{7} = 6435 \]
Legendre’s Identity

Let $\mu(n)$ be the largest integer exponent of the prime power $p^{\mu(n)}$ which divides $n!$

i.e. $n!$ is divisible by $p^{\mu(n)}$ and not $p^{\mu(n)+1}$

$$\mu(n) = \frac{n - \sigma}{p - 1}$$

$$\sigma = a_m + a_{m-1} + \cdots + a_0$$

$$(a_m a_{m-1} \cdots a_0)_p$$
Legendre’s Identity

• Take 5! (or 120)

  • Is it divisible by a power of 2?
    \[\mu(5) = \frac{5 - 2}{2 - 1} = 3\]
    \[n = 5\]
    \[\sigma = 2\]
    \[p = 2\]

  • Is it divisible by a power of 3?
    \[\mu(5) = \frac{5 - 3}{3 - 1} = 1\]
    \[p = 3\]
    \[\sigma = 3\]

  • Is it divisible by a power of 5?
    \[\mu(5) = \frac{5 - 1}{5 - 1} = 1\]
    \[p = 5\]
    \[\sigma = 1\]
How many Black Cells?

- Colour odd numbers black, even white
- How many cells in a specific row are black?
- How many cells in a specific triangle are black?
  - Depends on how many rows
- What proportion of the triangle is black?
- Allows a better understanding of the contradictory nature of fractals
The Cantor Set

Done to death...

How many in the row?

- Expand the row \((r)\) into its binary expansion
- For divisibility by 2

\[
r = c_0 + c_1 \cdot 2 + c_2 \cdot 2^2 + \cdots + c_m \cdot 2^m
\]

\[
h_2(r) = \prod_{i=0}^{m} (c_i + 1)
\]
How many in the row?

- Row 15: \[ r = 15 = (1111)_2 \]

\[
h_2(15) = (1 + 1) \cdot (1 + 1) \cdot (1 + 1) \cdot (1 + 1) = 2^4 = 16
\]
How many in the triangle?

- Needs:
  - How many in each row?
  - How many rows?

\[ h_2(r) = \prod_{i=0}^{m} (c_i + 1) + 2^k \text{ rows} \]

For \(2^k\) rows, there are \(3^k\) black cells.
What proportion?

- Needs:
  - Number of black cells
  - Total number of cells

\[
\frac{n(n + 1)}{2} \quad n = 2^k
\]

\[
Proportion = 2 \cdot \frac{3^k}{2^k(2^k + 1)}
\]
Significance?

- Limit?

\[
\lim_{k \to \infty} 2 \cdot \frac{3^k}{2^k(2^k + 1)} = 0
\]
Significance?
Summary

- The Basics
- Cellular Automata
- Divisibility Theorems
- Darts

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Back to Kummer’s Result

\[ n = a_0 + a_1 p + a_2 p^2 + \cdots + a_m p^m \]
\[ k = b_0 + b_1 p + b_2 p^2 + \cdots + b_m p^m \]

\[ p^\nu \text{ divides } \binom{n + k}{k} \]
\[ \nu = \mu(n + k) - \mu(n) - \mu(k) \]

\[ \binom{n + k}{k} = \frac{(n + k)!}{n!k!} \]
Back to Kummer’s Result

\[ c_p(n, k) = \mu(n + k) - \mu(n) - \mu(k) \]

\[ \epsilon_0 = \left[ \frac{a_0 + b_0}{p} \right] \quad \epsilon_i = \left[ \frac{a_0 + b_0 + \epsilon_{i-1}}{p} \right] \]

\[ \epsilon_{-1} = 0 \]

\[ c_p(n, k) = \sum_{i=0}^{\infty} \epsilon_i \]
Back to Kummer’s Result

\[ n + k = c_0 + c_1p + c_2p^2 + \cdots + c_m p^m \]

\[ c_i = a_i + b_i + \epsilon_{i-1} - \epsilon_i p \]

\[ \nu = \mu(n + k) - \mu(n) - \mu(k) \]

\[ = \frac{n + k - \sum_{i=0}^{\infty} c_i}{p - 1} - \frac{n - \sum_{i=0}^{\infty} a_i}{p - 1} - \frac{k - \sum_{i=0}^{\infty} b_i}{p - 1} \]

\[ = \frac{1}{p - 1} \left( \sum_{i=0}^{\infty} a_i + \sum_{i=0}^{\infty} b_i - \sum_{i=0}^{\infty} (a_i + b_i + \epsilon_{i-1} - \epsilon_i p) \right) \]

\[ = \frac{1}{p - 1} \left( 1 + \frac{\sum_{i=0}^{\infty} \epsilon_i p - \sum_{i=0}^{\infty} \epsilon_{i-1}}{p - 1} \right) \]

\[ = \frac{1}{p - 1} \left( 1 + \frac{\sum_{i=0}^{\infty} \epsilon_i p - \sum_{i=0}^{\infty} \epsilon_{i-1}}{p - 1} \right) \]

\[ = \frac{1}{p - 1} \left( \sum_{i=0}^{\infty} \epsilon_i p - \sum_{i=0}^{\infty} \epsilon_{i-1} \right) \]

\[ = \frac{1}{p - 1} \left( \sum_{i=0}^{\infty} \epsilon_i p - \sum_{i=0}^{\infty} \epsilon_i \right) \]

\[ \nu = \frac{1}{p - 1} \sum_{i=0}^{\infty} \epsilon_i (p - 1) \]

\[ \nu = \sum_{i=0}^{\infty} \epsilon_i \]

\[ \nu = c_p(n, k) \]
Limit proof

\[
2 \cdot \frac{3^k}{2^k (2^k + 1)} = 2 \cdot \frac{3^k}{4^k + 2^k}
\]

\[
= 2 \cdot \frac{3^k}{4^k (1 + (\frac{1}{2})^k)}
\]

\[
= 2 \cdot \left(\frac{3}{4}\right)^k \cdot \frac{1}{1 + (\frac{1}{2})^k} \quad \rightarrow 0 \quad as \quad n \rightarrow \infty
\]