Chaos Without God’s Dice:
An Exploration of Deterministic Systems

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Abstract. This paper will analyse further how chaotic behaviour arises in ostensibly simple deterministic systems. We will utilise binary representations to simplify analysis of the saw-tooth transformation’s chaotic properties. The notions of topological conjugacy and semi-conjugacy will be introduced to demonstrate that the quadratic iterator inherits the chaotic behaviour of the saw-tooth transformation. Finally the attempts to approximate chaotic orbits of deterministic systems will be shown to have surprising worth via the Shadowing Lemma

Keywords. deterministic chaos, quadratic iterator, topological conjugacy, shift operator, shadowing lemma

Introduction

The study of chaos has exploded in the last half century as science leapt at the challenge to understand those systems whose future is fundamentally shrouded in uncertainty. From the weather to those as simple as the double-pendulum, being governed by a set of constant and well-understood rules does not make the future any more fathomable. This inability to predict behaviour given estimates of initial conditions is the hallmark of the phenomenon we seek to study.

By definition, this sensitivity limits the reach of any conclusions we make with the aid of computer simulations. Each time a computer performs a calculation using its floating point arithmetic, truncation occurs. Although the effect is negligible in every other case, chaos is always unforgiving, with errors inevitably growing to the size of the measurement as the system evolves. Fixed points move about and periodic points never return in computer simulations of the systems we are interested in. Fig. 1 below demonstrates how this property was discovered in the Lorenz equation. In light of this, we are motivated to search for properties of chaos that we can confirm analytically with resorting to misleading computer simulations.

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Distinguishing chaos from the complex but predictable is not always self-evident. The most intuitive and familiar understanding of what chaos is lies in its sensitivity to initial conditions. The definition we use was formulated by Devaney and has been widely accepted in the field. He adds two characteristics to complement our intuitive base:

1. Sensitive Dependence on Initial Conditions
2. Mixing
3. Density of Periodic points

We examine each of these individually, using the quadratic iterator

\[ x_{n+1} = 4x_n(1 - x_n) \]

as an example.

1.1. Independence of the conditions?

Although originally proposed as three distinct criterion, it can be shown that if a system has the properties of mixing and density of periodic points, it must have sensitive dependence on initial conditions.\(^2\) We explore all 3 properties nonetheless to obtain a feel for this chaotic behaviour.

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1.2. Sensitivity

A butterfly flaps its wings in Brazil and causes a tornado in Texas. A minute error grows bigger and bigger until it renders the output irrelevant to its original conditions. *Sensitive dependence on initial conditions* means that the smallest difference in starting value will eventually cause the output to deviate after repeated iterations. For the quadratic iterator consider a starting interval of arbitrarily small size and observe that repeated iterations cause the interval to grow as small initial differences are magnified as seen in Fig. 2.

1.3. Mixing

The *mixing* property is satisfied when any arbitrarily small interval contains a point which eventually hits any arbitrary interval in space. We see this between the intervals $I$ and $J$ for the quadratic iterator in Fig. 3.

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3Inspired by a title given by Philip Merilees to a talk given by the chaos-theory pioneer Edward Lorenz at the American Association for the Advancement of Science in 1972
1.4. Periodic Points

A prerequisite of Devaney’s Chaos is that periodic points be dense on the domain. This means that we can find a periodic point arbitrarily close to any point we like in the domain. It is quite easy to confirm that certain points such as $\sin^2\left(\frac{\pi}{7}\right)$ are periodic for the quadratic iterator and an orbit for such a point is mapped above in Fig. 4.

The decidedly non-periodic result reminds us again of the limit of computation simulations to confirm chaotic behaviour and prompts a return to an analytical approach to confirm these properties exist.

2. The Saw-Tooth Function

We begin our search for analytical demonstrations of chaos with one of the simplest functions displaying chaos, the Saw-Tooth function.

$$\frac{\text{Frac}}{}(2x) = S(x) = \begin{cases} 2x & \text{if } 0 < x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

2.1. A Less Ominous Form - The Shift Operator

The Saw-Tooth function is also known as the shift operator due to its equivalent operation in binary form.\(^4\)

$$x = a_12^{-1} + a_22^{-2} + a_32^{-3} + a_42^{-4} + \ldots = 0.a_1a_2a_3a_4\ldots$$

$$\rightarrow S(x) = 0.a_2a_3a_4\ldots$$

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\(^4\)We must choose the non-repeating representation of numbers if there is one, such as $\frac{1}{3} = 0.1$ rather than 0.01. The shift-operator is also not defined for $x=1$, so we restrict our investigation to [0,1)
We harness this form of the transformation to demonstrate analytically Devaney’s three criterion holding.

2.2. Sensitivity

For $a_i, b_i$ either 1 or 0 and $a_i \neq b_i$, we take two close points which agree for their first $k$ binary digits. This restricts them to being at most $2^{-k}$ apart for arbitrary integer $k$.

$$x = 0.a_1a_2a_3a_4...a_k{1}a_{k+2}...$$
$$y = 0.a_1a_2a_3a_4...a_k{1}b_{k+2}...$$

We perform $k$ iterations on both points to obtain:

$$S(x) = 0.a_2a_3a_4...$$
$$S(y) = 0.a_2a_3a_4...$$
$$\vdots$$
$$S^k(x) = 0.a_{k+1}a_{k+2}a_{k+3}...$$
$$S^k(y) = 0.b_{k+1}b_{k+2}b_{k+3}...$$

The points have completely diverged and after the first $k$ iterations their behaviour will bear no resemblance to each other, demonstrating sensitivity.
2.3. Mixing

For any interval $I$ with width $2^{-k}$ we may define its points as the numbers beginning with the binary expansion:

$$x_I = 0.a_1a_2a_3a_4...a_k$$

Now suppose we wish to reach a point $b = 0.b_1b_2b_3b_4...b_j$. In the interval $I$ there exists a point with binary representation:

$$x = 0.a_1a_2a_3...a_kb_1b_2b_3...b_j$$

$$S^k(x) = 0.b_1b_2b_3...b_j$$

Evidently, after $k$ iterations, the point $x$ will reach the point $b$ exactly on its orbit, proving that from any arbitrarily small interval, there exists a point which will eventually reach any other desired point in the interval. This is demonstrated for two starting points in Fig. 6.

2.4. Periodic Points

Given a non-periodic point say $x = 0.a_1a_2a_3...a_k$, can we find a periodic point arbitrarily close?
Figure 7. A cycle of period 4 in the Saw-Tooth

Define the point:

\[ x_p = 0.a_1a_2a_3...a_k \]

Now due to the repeating nature of the binary expansion, we can see that the point is clearly periodic with length \( k \), after \( k \) iterations the shift operator will have taken it through the complete period, only to begin again. In fact, we can make this point even closer to our original \( x \) by letting it match the zeros following \( a_k \) for as long as we like. Let

\[ x_p = 0.a_1a_2a_3...a_k000...0 \]

This holds for any non-periodic point with finite decimal expansion, which is a dense set on \([0,1)\). Fig. 7 demonstrates this periodic behaviour in the saw tooth.

3. Topological Conjugacy

We introduce the notion of topological conjugacy with the aim of extending what we now know about the shift operator to other transformations.

3.1. Definition

Define \( X \) and \( Y \) as two subsets of the real line and let \( f \) and \( g \) be transformations, \( f : X \to X \) and \( g : Y \to Y \). Then \( f \) and \( g \) are said to be topologically conjugate, provided \( f \) and \( g \) are continuous and there is a homeomorphism \( h : X \to Y \) such that the functional equation holds:

\[ h(f(x)) = g(h(x)) \]
Homomorphisms are continuous functions which are bijective with continuous inverses. Fig. 8 represents Equation 1 in a visual sense, where $\alpha$ is the homeomorphism $h$.

![Visual Representation of the Functional Equation 1](image)

Figure 8. Visual Representation of the Functional Equation 1

3.2. An Example in Quadratics

A simple example of this is the relationship between quadratic polynomials. They are all topologically conjugate to $g(x) = x^2 + c$ for some $c$ by an affine linear map $h(x) = mx + n$. For example if $f(x) = 4x(1 - x)$, $g(x) = x^2 - 2$ there exists the homeomorphism $h(x) = -4x + 2$ such that:

$$
g(h(x)) = g(-4x + 2)$$

$$= (16x^2 - 16x + 4) - 2$$

$$= 16x^2 - 16x + 2$$

$$h(f(x)) = h(4x(1 - x))$$

$$= -4(4x - 4x^2) + 2$$

$$= 16x^2 - 16x + 2$$

$$\therefore g(h(x)) = h(f(x))$$

3.3. Use in Chaos

Mixing and Density of periodic points are both topological properties, meaning if they are present in one transformation they are present in all other transformations that are topologically conjugate to them. As sensitivity to initial conditions is implied by the satisfaction of the first two of Devaney’s properties, showing a relationship of topological conjugacy between two systems where one is known to be chaotic implies that the other one is, without needing to run a single iteration.

3.4. Semi-Conjugacy and Inherited Chaos

We now explore the weaker relationship of topological semi-conjugacy. This occurs when we drop the requirement for $f$ and $g$ to be continuous and merely require that $h$ is
surjective and the functional equation holds. Despite the lack of equivalence (one function being topologically semi-conjugate to another does not imply the latter is topologically semi-conjugate to the first), the two properties we are interested in are still retained across topologically semi-conjugate transformations. We prove this for periodic points of length 3 (the proof for length \( k \) can be completed by induction). Given \( f(h(x)) = h(g(x)) \) and \( g^3(x_p) = x_p \) where \( x_p \) is a periodic point of length 3, we aim to show that \( h(x_p) \) is a periodic point of length 3 in \( f \):

\[
\begin{align*}
  f^3(h(x_p)) &= f^2[f(h(x_p))] \\
  &= f^2(h(g(x_p)) \\
  &= f[f(h(g(x_p)))] \\
  &= f[h(g(x_p))] \\
  &= f[h(g^2(x_p))] \\
  &= h(g^2(x_p)) \\
  &= h(x_p)
\end{align*}
\]

\[
\therefore f^3(h(x_p)) = h(x_p)
\]

4. The Quadratic Iterator

Due to the lack of a simple expression for the \( k \)-th iterate of the quadratic iterator, we are forced to search for other means to demonstrate analytically its chaotic behaviour. As we have discovered, the shift-operator has an accessible expression for the \( k \)-th iterate. Further we know that the relationship of topological semi-conjugacy transfers chaotic properties from one transformation to another. With this in mind, we seek to show that the quadratic iterator is topologically semi-conjugate to the shift-operator.

4.1. Relationship with the Shift Operator

Let \( f(x) = 4x(1-x) \), \( g(x) = S(x) \). Our conjugation is \( h(x) = \sin^2(\pi x) \)

\[
h(g(x)) = \begin{cases} 
  \sin^2(2\pi x) & \text{if } 0 < x < \frac{1}{2} \\
  \sin^2((2x - 1)\pi) & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
\]

But

\[
\begin{align*}
  \sin^2((2x - 1)\pi) \\
  &= |\sin(\pi - 2\pi x)|^2 \\
  &= \sin^2(2\pi x) \\
  \therefore h(g(x)) &= \sin^2(2\pi x)
\end{align*}
\]
\[ f(h(x)) = 4\sin^2(\pi x)(1 - \sin^2(\pi x)) \]
\[ = [2\sin(\pi x)\cos(\pi x)]^2 \]
\[ = [\sin(2\pi x)]^2 \]
\[ = h(g(x)) \]

Therefore the quadratic iterator inherits the topological properties of mixing and density of periodic points, which necessarily imply sensitivity and we are able to conclude that the quadratic iterator is chaotic. Further the conjugacy of the quadratic iterator to \( g(x) = x^2 - 2 \) implies that its behaviour is chaotic too.

4.2. Finding Periodic Points of Any Length

As periodic points are preserved under the topological conjugacy and given how easy it was to generate periodic points for the shift-operator, we simply pass these through the transformation \( h(x) = \sin^2(\pi x) \) to generate period points for the quadratic iterator. For example the general form of periods of length 3 for the shift operator is:

\[ x_{p3} = 0a_1a_2a_3a_2a_3a_1a_2a_3... = 0.a_1a_2a_3 \]
\[ = \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \frac{a_1}{16} + ... \]
\[ = \frac{(4a_1 + 2a_2 + a_3)}{7} \]

As \( a_i \) is 1 or 0, we know periodic points exist for the quadratic iterator at \( \sin^2(\frac{\pi}{7}) \), \( \sin^2(\frac{2\pi}{7}) \), \( \sin^2(\frac{3\pi}{7}) \) of length 3, which is easily confirmed by exercise.

5. The Shadowing Lemma: A Surprising Consolation

Although this paper began with a lament at the inability of computers to aid our search for prediction in deterministic chaos, it turns out things are not as bad as they seem. Consider a deterministic system such as the ones we have discussed above. As it evolves through repeated iterations, the computed orbit will necessarily stray from the true orbit due to the approximation falling prey to the unavoidable sensitivity. Eventually, our computed orbit has deviated completely from the behaviour of the original as shown in the upper half of Fig. 9. However the Shadowing Lemma states that if the errors \( \epsilon \) of the computed orbit at each iteration \( k \) are bounded, that is:

\[ |\epsilon_k| \leq \epsilon \]

then lying in the "shadow" of size \( \epsilon \) of the computed orbit is a true orbit for the system which traces its approximate path as seen in the lower half of Fig. 9. This result is remarkable in the sense that the richness of these chaotic systems provides not just
the possibility of reaching any interval starting from any interval, but the possibility of following any path we like to get there. Note that the only condition was the upper limit on the errors $\epsilon_k$, their distribution can be defined arbitrarily to produce a path that looks as simple or complex as we like.

The striking consequence is that where the lemma holds, a deterministic model will support almost any prediction, meaning any set of suitably constrained data could be construed to come from a given system. This hints at the problem of how it is all but impossible to deduce the characteristics of a chaotic system given its output. Nonetheless, the Shadowing Lemma gives us comfort that our computer simulations are displaying real behaviour of the system, even if it is not the specific orbit we were interested in.

**Conclusion**

Our inaccurate computer simulations have shown us how difficult it is to prove chaotic behaviour is occurring computationally. Despite the difficulty of explicitly finding $k$-th iterates of deterministic systems such as the quadratic iterator, we have demonstrated how the notion of topological conjugacy can aid our understandings of chaotic behaviour by exploiting the equivalence between easy to iterate systems and more complex ones. Finally, the Shadowing Lemma delivers some respite from all the gloom and doom about using computers to study chaos, but still does not get us closer to any the sought-after predictability. It is fruitless after all, as nearly knowing where we are is never good enough in the realm of chaos.
References


