The Mandelbrot set- The most maesthetic object

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What is it
Introductions to Mandelbrot and his pretty picture

Critical Hit!
Critical orbit of quadratic iterator
Critical Orbit for transformed quadratic iterator
Connectedness of Julia Sets

Buds
The Big Blob
The Second Biggest Blob

Similarity
Magnification and asymptotic self-similarity
Similarity to Julia Sets

Fun Fractal Facts(optional)
Aesthetics and beauty
Tribute
### Mr Benoit B. Mandelbrot

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- Discovered the Mandelbrot set
- Has an asteroid named after him
Define $M$

The Mandelbrot Set ($M$)

The Mandelbrot set, which we shall denote as the set $M$, may be defined in the following ways:

1. The set in $\mathbb{C}$ such that the corresponding Julia Set is connected.
   
   $$M := \{ c \in \mathbb{C} \mid J_c \text{ is connected} \}$$

2. The set in $\mathbb{C}$ such that the sequence $0 \rightarrow c \rightarrow c^2 + c \rightarrow \cdots$ doesn’t escape to infinity.

   $$M := \{ c \in \mathbb{C} \mid 0 \rightarrow c \rightarrow c^2 + c \rightarrow \cdots \text{ remains bounded} \}$$
Our familiar Quadratic iterator:

\[ x \rightarrow ax(1 - x) \]

Topologically conjugate to

\[ z \rightarrow z^2 + c \ (z, c \in \mathbb{R}) \]

**BUT HOW?**

For \( c \leq 1/4 \) We can transform one to another using

\[ x_n = \frac{1}{2} - \frac{z_n}{a}, \ a = 1 + \sqrt{1 - 4c} \]
Backwards Graphical Iteration

1. Begin with $Q_a^{(0)} \supset P_a$

We will begin this demonstration with $g(x) = ax(1 - x)$ on $\mathbb{R}$ and move on to $z \rightarrow z^2 + c$ for $\mathbb{R}$ and then $\mathbb{C}$
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Backwards Graphical Iteration

1. Begin with $Q_a^{(0)} \supset P_a$
2. Apply a 'reverse cobwebbing' process
3. Profit.

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The process maintains an interval for $a = 3.5$ but generates a Cantor set for $a = 4.5$.
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Criteria:
Criteria: vertex goes above $[0, 1] \times [0, 1]$
Critical point and critical value $(x, y)$ occur at:

$$(x, y) = \left(\frac{1}{2}, \frac{a}{4}\right)$$

i.e. one solution for $g(x) = ax(1 - x)$
Critical Orbit

- Critical orbit is the orbit of $x_0 = 1/2$
- $\frac{1}{2} \rightarrow \frac{a}{4} \rightarrow \frac{4a^2-a^3}{16} \rightarrow \ldots$
- if $a \leq 4 : \text{critval} = \frac{a}{4} \leq 1$, Critical orbit remains in $[0, 1]$
- if $a > 4 : \text{critval} = \frac{a}{4} > 1$, Critical orbit escapes to $-\infty$, $P_a$ is a Cantor set.
So we just found the Critical Orbit for $g(x) = ax(1-x)$
What about $z_{n+1} = z_n^2 + c \quad \{z, c \in \mathbb{R}\}$?

**Transforming**

Using our transformation from before:

$$x_n = \frac{1}{2} - \frac{z_n}{a}, \quad a = 1 + \sqrt{1 - 4c}$$

We get a concave up parabola with vertex along the y-axis.
Let us apply the same backwards iteration
• The essential square → determined by top right hand corner.
• This is the positive solution to \( z^2 + c = z \), \( z = \frac{1+\sqrt{1-4c}}{2} \) (for \( c < 1/4 \))
• Critical orbit: initial point \( Z_0 = 0 \)
• \( P_c \subset [-a/2, a/2] \) if \( \min f(z) \) is within the essential square
• \( 0 \to c \to c^2 + c \to \cdots \)
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• \( 0 \rightarrow c \rightarrow c^2 + c \rightarrow \cdots \)
• Looks familiar?
**connectedness on** $\mathbb{R}$

Vertex touches essential square at $c = -a/2$

When $c$ is large all orbits escape

$P_c$ is connected on $[-2, 1/4]$ in $\mathbb{R}$
extending to $\mathbb{C}$

**Binary Decompositions**

Source: Peitgen, Jurgens and Saupe, ”Chaos and fractals: new frontiers of science,” $2^{nd}$ edition
Steps:

1. Take a level set $L^k_c$

2. Split into two

$$L^k_c(0) = \{ z \in L^k_0 | Imz \geq 0 \}$$
$$L^k_c(1) = \{ z \in L^k_0 | Imz \leq 0 \}$$

3. The next sets $L^k_c(00)$, $L^k_c(01)$, $L^k_c(10)$, $L^k_c(11)$ are pre-images of $L^k_c(0)$ and $L^k_c(1)$ and appear in anticlockwise order from the positive real axis.

4. $L^k_c(1b_n \cdots b_1)$ is a reflection about the origin of $L^k_c(0b_n \cdots b_1)$
- Field Lines can be read off the the cells they pass through
- $\frac{1}{3}$, $\frac{2}{3}$ shown in diagram
- dynamics of $c=0$ equivalent to any $z \to z^2 + c$ (shown by Douady and Hubbard)

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Let us state the following:

1. Cell 0110 has only 1 pre-image. Due to symmetry, it must contain the point 0.
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The fate of Critical orbits

Let us state the following:

1. Cell 0110 has only 1 pre-image. Due to symmetry, it must contain the point 0.
2. $L_{c}^{m-1}$ has a figure 8 shape which disconnects two parts of $P_{c}^{m-2}$.
3. $P_{c}$ decomposes into at least two parts
4. The critical orbit $0 \rightarrow 0110 \rightarrow L_{c}^{m+1}$ must diverge.
"Heart" Shaped Blob (on $\mathbb{R}$)

Properties:

- intersects $[-0.75, 0.25] \in \mathbb{R}$
- super attractive at 0
- attractive fixed point at $\frac{1-\sqrt{1-4c}}{2}$
- What about the other fixed point?
In fact:

- Set of $c \in \mathbb{C}$ such that one of the two fixed points of $z \to z^2 + c$ is attractive
- Indifferent fixed point ($\left| \frac{df}{dz} \right| = 1$) on the boundary
- What about the other fixed point?
On the boundary of big blob

- $z^2 - z + c = 0$
- $2z = re^{i\phi}, \ (r \geq 0, \ 0 \leq \phi < 2\pi)$
- $(\frac{re^{i\phi}}{2})^2 - \frac{re^{i\phi}}{2} + c = 0$
- solving for $c$, we get $c = \frac{1}{2} re^{i\phi} - \frac{1}{4} r^2 e^{2i\phi}$
- for $c = x + iy, \ r = 1$ we have
  - $x = \frac{\cos\phi}{2} - \frac{\cos2\phi}{4}$
  - $y = \frac{\sin\phi}{2} - \frac{\sin2\phi}{4}$
A bud occurs at each of the parameters
\[ \phi = \frac{2\pi}{k}, \quad k = 2, 3, 4, 5, 6 \ldots \]

Period of cycles within these buds given by \( k \)
Perfect Circle
Perfect Circle

- Centred at $-1 + 0i$
- Neither fixed point attractive
- Super attractive period 2 orbit $-1 \rightarrow 0 \rightarrow -1 \cdots$

Do periods (rather than simply fixed points) satisfy the derivative criterion?

**Proof.**
-1, 0 periodic for $z \rightarrow z^2 - 1$
Iterate twice: $z \rightarrow z^4 - 2z^2$
Derivative $= 4z^3 - 4z$
Subbing in -1 and 0 gives a derivative of 0.
Magnification

Recall: an image $I$ is self similar if

$$I = \omega_1(I) \cup \omega_2(I) \ldots \cup \omega_n(I)$$

where $\omega_i$ are similarity transformations.

Scaling factor: $\rho$, $\rho I = \{\rho z | z \in I\}$

An object that satisfies $I = \rho I$ extends to infinity.

In the neighbourhood of a point:

If $\exists \epsilon > 0 | D_r(z) \cap I = D_r(z) \cap \rho I \ \forall r < \epsilon$ then $I$ is self-similar in $z$. 
Magnification

Can $\rho$ be complex?

A complex factor $\rho$ simply corresponds to a rotation by $arg \rho$ and a magnification by $|\rho|$.

Logarithmic spirals are self-similar in 0. (of the form $\log r = a\phi$)

"Logarithmic Spiral Pylab" by Morn the Gorn.
Asymptotic Self-similarity at a point

We begin by translating \( Z_0 \) to 0 by considering similarity at point \( z \) by \( I - z_0 \)

**Definition**
There exists:

1. A complex scaling factor \( \rho \) with \( |\rho| > 1 \)
2. A small radius \( r > 0 \)
3. A limit object \( L \) which is self-similar at the origin

Such that the following holds:

\[
\lim_{n \to \infty} D_r(0) \cap p^n(I - z_0) = L \cap D_r(0)
\]
J, M, and similarity

• Unfortunately, we cannot say that M or $J_c$ are self-similar at any point.
• However they are asymptotically self similar!
• i.e. the limit of the scaling procedure on these sets converges to an object which is self-similar at a point.
Misiurewicz Points

• Definition: a point $c$ such that the critical orbit is pre-periodic (but not periodic) under the map $f_c$.
• i.e. $\exists m \geq 1$ and $p \geq 1$ s.t. $z_m = z_{m+p}$
• All such points are in $M$ by definition
• Exist on the boundary of $M$
• Types:
  1. branch points
  2. end points
  3. non-branch points
Misiurewicz Points

Properties + Tan Lei’s Theorem

• Associated $P_c = J_c$ = has no interior (is a dendrite).
• Misiurewicz points are dense on the boundary of $M$
• $M$ and $J_c$ are asymptotically similar at Misiurewicz points.
• The associated limit objects differ only by a scaling and a rotation i.e.

$$L_M = \lambda L_j, \lambda \in \mathbb{C}$$
Example: the Julia set for $c = i$

- $i \rightarrow -1 + i \rightarrow -i \rightarrow -1 + i \rightarrow -i \rightarrow -1 + i \ldots$
- A popular dendrite is formed.

Source:
http://www.emba.uvm.edu/~jxyang/teaching/Math266
Figure: Julia and Mandelbrot sets magnified at $c \approx -0.1011 + 0.9563i$. 
Dave Boll (1991) discovered an astounding relationship between the number of iterations $N$ points of the form $-0.75 + \epsilon i$ ($\epsilon$ small) took to escape.

$$\lim_{\epsilon \to 0} \text{Iterations} \times \epsilon = \pi (\pm \epsilon)$$
WHAT HAPPENS NEXT WILL BLOW YOUR MIND: if two buds have periods $p$ and $q$, the largest bulb between them on the cardioid has a period of $p+q$!
ARTS STUDENTS HATE HIM: The number of spokes at the top corresponds to the period of the bud! (periods 3, 4, 5, 7 shown)
FIRST YOU’LL BE SHOCKED, THEN YOU’LL BE INSPIRED:
The periodic windows correspond to the secondary Mandelbrot sets along the tip of M.
Is the fibonacci sequence in $M$ 
(YES)(sort of)
A quote

"You’re one badass f***ing fractal”

-Jonathan Coulton, on the Mandelbrot set, in his song *Mandelbrot Set*
Thank you Mr Mandelbrot

Legend has it, if you magnify/rotate at his middle name something interesting appears...
Thank you Textbook

Pietgen, Jurgens and Saupe:
Thank you Robby