The University of Sydney
School of Mathematics and Statistics

Assignment

MATH2962: Real and Complex Analysis (Advanced) — Semester 1, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
Lecturer: Florica Cîrstea

Due on **Friday, 6 May** by **5pm** in Carslaw **Room 719**
(Slide under the door when locked).
Late assignments are not accepted without *prior arrangement* well before the deadline!
You must attach the signed cover-sheet to the front of your assignment (see page 3)!

1. For $n \in \mathbb{N} \setminus \{0\}$, we define c_n as follows

$$c_n := \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n.$$

(a) Show that (c_n) is a decreasing sequence of positive numbers. *2 Marks*

(b) For $n \in \mathbb{N} \setminus \{0\}$, we also define *1 Mark*

$$b_n := \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{2n}.$$

Prove that $b_n = c_{2n} - c_n + \ln 2$.

(c) Using (a) and (b), show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges with the limit equal to $\ln 2$. *1 Mark*

2. (a) If the complex power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, what is the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} a_n z^{2n}$? Justify your answer. *1 Mark*

(b) For $n \in \mathbb{N} \setminus \{0\}$, we define $a_n = 1/n$ for n even and $a_n = 1/n^2$ for n odd. Prove that the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ diverges although $a_n > 0$ for every $n \geq 1$ and $a_n \to 0$ as $n \to \infty$. Does this contradict Leibniz test? Justify your answer. *2 Marks*

(c) Consider the series

$$\sum_{n=1}^{\infty} \frac{\cos(\frac{n\pi}{3})}{\sqrt{n}}.$$

(i) Prove or disprove that this series is absolutely convergent. *1 Mark*

(ii) Using the sequence of partial sums, prove that the series converges. *3 Marks*

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3. Let $(x_n)_{n\geq 0}$ be a non-zero sequence in $\mathbb{R}$. Assume that the following limit

$$\beta := \lim_{n\to\infty} n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right) \quad \text{exists in } (1, \infty]. \tag{1}$$

Let α be fixed with $1 < \alpha < \beta$.

(a) Prove that there exists $n_\alpha \geq 1$ such that *2 Marks*

$$\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{\alpha}{n} \quad \text{for every } n \geq n_\alpha.$$

(b) Using (a), show that *1 Mark*

$$(k-1)|x_k| - k|x_{k+1}| \geq (\alpha - 1)|x_k| \quad \text{for every } k \geq n_\alpha.$$

(c) Using (b), prove that the series $\sum_{n=0}^{\infty} x_n$ is absolutely convergent. *3 Marks*

(d) Assume that none of the numbers A, B and C is a negative integer or zero. Using the conclusion of (c), prove that the **hypergeometric series** *3 Marks*

$$\frac{AB}{1!C} + \frac{A(A+1)B(B+1)}{2!C(C+1)} + \frac{A(A+1)(A+2)B(B+1)(B+2)}{3!C(C+1)(C+2)} + \dots$$

is absolutely convergent for $C > A + B$.

Remark 1. The harmonic series is known to diverge, but the convergence of the sequence c_n (defined in Question 1) was observed by Euler. The limit of the sequence, $\gamma := \lim_{n\to\infty} c_n$, is called **Euler's constant** and $\gamma \approx 0.5772156649...$ Although computers have calculated over two trillion decimal places, it is a famous unsolved problem whether γ is a rational or irrational number. The prevailing opinion is that γ is irrational, but no proof has yet been found.

Remark 2. Question 3 is the limiting form of **Raabe's Test**, which is *frequently useful when the ratio test cannot be applied* such as when $\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = 1$. Note that (1) is automatically satisfied with $\beta = \infty$ if $\limsup_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| < 1$ and, in this case, the absolute convergence of the series $\sum_{n=0}^{\infty} x_n$ can be obtained from the ratio test.

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Assignment Cover Sheet

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Family Name ....

Given Names .... **SID** ....

Some collaboration between students on assignments is encouraged, since it can be a real aid to understanding. Thus it is legitimate for students to discuss assignment questions at a general level, provided everybody involved makes some contribution. However, students should produce their own individual written solution. Copying someone else's work is plagiarism, and is unacceptable. The University may impose severe penalties in cases where plagiarism is detected.
I certify that:

- I have read and understood the *University of Sydney Student Plagiarism: Coursework Policy and Procedure* at http://www.maths.usyd.edu.au/u/UG/Plagiarism.pdf.
- this assignment is all my own work, and that no part of this assignment has been copied from another person.
- I have not allowed my work to be copied by another person.

Signature .... **Date** ....

This part to be completed by the marker:

Grand total out of 20 ....