Solution to Question 1 from $\boxed{\text{Practice Class Week 9}}$

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Week 10

$\frac{1}{f}$ is uniformly continuous on $A$ if and only if

$\forall \varepsilon > 0, \ \exists \delta = \delta(\varepsilon) > 0$ s.t. $\left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| < \varepsilon$

$\forall x, y \in A$ s.t. $|x - y| < \delta$.

Fix $\varepsilon > 0$ arbitrary. Evaluate

$$\left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| = \left| \frac{f(y) - f(x)}{f(x) f(y)} \right| = \frac{|f(x) - f(y)|}{|f(x)| |f(y)|}$$

We know that $|f(x)| \geq k > 0 \quad \forall x \in A$. Hence

$$\frac{1}{|f(x)|} \cdot \frac{1}{|f(y)|} \leq \frac{1}{k^2}, \quad \forall x, y \in A.$$

Therefore, we have

(1) $$\left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| \leq \underbrace{\frac{|f(x) - f(y)|}{k^2} \boxed{\overset{?}{<} \varepsilon}}$$

hypothesis

Since $f$ is uniformly continuous on $A$, there exists $\delta = \delta(\varepsilon) > 0$

s.t. $|f(x) - f(y)| < k^2 \varepsilon, \quad \forall x, y \in A$ with $|x - y| < \delta$.

(2)

From (1) and (2), it follows that

$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

$$\left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| < \varepsilon, \quad \forall x, y \in A \text{ with } |x-y| < \delta.$$

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Solution to Q2) from $\boxed{\text{Practice class Week 9}}$

(i) $f(x) = x$ is unif. cont on $\mathbb{R}$

$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

$$|x-y| < \varepsilon, \quad \forall x, y \in \mathbb{R} \text{ with } |x-y| < \delta.$$

true: take $\delta = \varepsilon$.

(ii) $g(x) = \sin x$ is unif. cont. on $\mathbb{R}$.

$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

$$|\sin x - \sin y| < \varepsilon, \quad \forall x, y \in \mathbb{R} \text{ with } |x-y| < \delta.$$

By the mean value theorem

$$|\sin x - \sin y| = |x-y| \cdot |\cos c| \text{ for some } c \text{ between } x \text{ and } y.$$

$|\sin x - \sin y| \leq |x-y| < \varepsilon$ for every

$x, y \in \mathbb{R}$ with $|x-y| < \delta$ $\quad \delta = \varepsilon$.

Hence,

$\forall \varepsilon > 0, \ \exists \delta = \varepsilon > 0$ s.t.

$|g(x) - g(y)| = |\sin x - \sin y| < \varepsilon, \quad \forall x, y \in \mathbb{R}$ with $|x-y| < \delta$.

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Q2) iii) We prove that $f(x)g(x)$ is <u>not</u> uniformly cont. on $\mathbb{R}$.

We assume by contradiction that $f(x)g(x) = x \sin x$ is unif. <u>cont. on $\mathbb{R}$</u>. Hence,

$$(3) \begin{cases} \forall \varepsilon > 0, \ \exists \delta = \delta(\varepsilon) > 0 \text{ s.t.} \\ \quad |x \sin x - y \sin y| < \varepsilon \\ \forall \underline{x, y \in \mathbb{R}} \text{ with } |x-y| < \delta. \end{cases}$$

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$|x \sin x - y \sin y| = |x-y| \cdot |\sin c + c \cos c|$

for some $c$ between $x$ and $y$.

Let $y = x + h$ with $\underbrace{h > 0}_{\text{fixed}}$ such that $h = |h| < \delta$.

From (3) we have

$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) > 0$ s.t.

$$D = \underbrace{|x \sin x - (x+h) \sin (x+h)|}_{\forall x \in \mathbb{R}} < \varepsilon$$

Evaluate $D = |x \sin x - (x+h)(\sin x \cosh + \cos x \sinh)|$

$$= |\underbrace{x \sin x - x \sin x \cosh - x \cos x \sinh}_{} - h \sin x \cosh - h \cos x \sinh|$$

this contains $x$ which can be made as large as we want

$$= |\overbrace{x}^{\sim} (\sin x - \sin x \cosh - \cos x \sinh)| - \overbrace{h (\sin x \cosh + \cos x \sinh)}^{\text{is bounded}}|$$

To get a contradiction, our aim is to prove that $D$ can be made large (as large as we want).

$$D \geq |x (\sin x - \sin x \cosh - \cos x \sinh)| - |h(\sin x \cosh + \cos x \sinh)|$$

$$D \geq |x| \cdot \underbrace{|\sin x (1-\cos h) - \cos x \sin h|} - M$$

where $M > 0$ s.t. $|h(\sin x \cosh + \cos x \sin h)| \leq M$.

Take $0 < h < \delta$ very small such that $\sin h \neq 0$.

Let $x_n$ be a sequence which goes to $\infty$ as $n \to \infty$.

$$x_n = 2\pi n \to \infty \quad \text{as } n \to \infty.$$

$$|x_n| \cdot |\underbrace{\sin x_n}(1-\cos h) - \underbrace{\cos x_n} \sin h|$$
$$= 2\pi n \, |0 - \sin h| = 2\pi n \, |\sin h| \to \infty \quad ? \text{ as } n \to \infty$$

Hence, $D \to \infty$ as $n \to \infty$.

Prove that

**Ex 3)** $f(x) = \cos\frac{1}{x}$ is <u>not uniformly continuous</u> on <u>$(0,1)$</u>.

$\boxed{\text{Solution}}$

We assume that $f$ is uniformly continuous. Hence

$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) > 0$ s.t.

$$\left|\cos\frac{1}{x} - \cos\frac{1}{y}\right| < \varepsilon$$

$\forall \underline{x, y \in (0,1)}$ s.t. $|x - y| < \delta$.

To get a contradiction this time we look for two sequences of positive numbers $x_n$ and $y_n$, both converging to $0$ as $n \to \infty$

(since $\cos\frac{1}{0^+} = \cos\infty$ does not exist)

such that $\left|\cos\frac{1}{x_n} - \cos\frac{1}{y_n}\right| \not\to 0$ as $n \to \infty$.

$x_n = \dfrac{1}{2\pi n}$ ; $\cos\frac{1}{x_n} = \cos(2\pi n) = \cos 0 = 1$.

$y_n = \dfrac{1}{\frac{\pi}{2} + 2\pi n}$ $\quad \cos\frac{1}{y_n} = \cos\left(\frac{\pi}{2} + 2\pi n\right) = \cos\frac{\pi}{2} = 0$.

Since $|x_n - y_n| \to 0$, we have $\forall \delta > 0$, $\exists n_\delta \geq 1$ s.t.

$$|x_n - y_n| < \delta, \quad \forall n \geq n_\delta.$$

For $\varepsilon \in (0, \frac{1}{2})$ fixed, we cannot have

$$\left( 1 \leftarrow \text{as } n \to \infty \right) \left| \cos\frac{1}{x_n} - \cos\frac{1}{y_n} \right| < \varepsilon \quad \blacksquare$$

since

$$\left| \cos\frac{1}{x_n} - \cos\frac{1}{y_n} \right| \to 1 \text{ as } n \to \infty.$$