Practice questions (Week 9)

A. uniform continuity

Q1) Let f be a uniformly continuous function on $A \subseteq \mathbb{R}$ such that

$$|f(x)| \geq k > 0 \quad \forall x \in A.$$

Show that $\frac{1}{f}$ is uniformly continuous on A.

Q2) If $f(x) = x$ and $g(x) = \sin x$, show that both f and g are uniformly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.

Q3) Prove that if f and g are each uniformly continuous on $\mathbb{R}$, then the composite function $f \circ g$ is uniformly continuous on $\mathbb{R}$

B: Pointwise / Uniform convergence

Q4) For any $n \geq 1$, we define

$$f_n : [0,\infty) \to \mathbb{R} \text{ by } f_n(x) = \frac{x^n}{1+x^n}.$$

(a) For every $x \geq 0$, evaluate

$$\lim_{n\to\infty} f_n(x) = f(x).$$

(b) Show that if $0 < b < 1$, then f_n converges uniformly on $[0,b]$.

(c) Show that f_n does not converge uniformly on $[0,1]$.

Q5) For any $n \geq 1$, we define

$$f_n(x) = \frac{\sin(nx)}{1+nx} \quad \text{for } x \geq 0.$$

(a) For every $x \geq 0$, evaluate $\lim_{n\to\infty} f_n(x) = f(x)$.

(b) Show that if $a > 0$, then f_n converges uniformly on $[a,\infty)$.

(c) Show that f_n does not converge uniformly on $[0,\infty)$.

Q6) For ~~every~~ $n \geq 1$, we define

$$f_n : [0,\infty) \to \mathbb{R}, \quad f_n(x) = x^2 e^{-nx}$$

(a) For every $x \geq 0$, evaluate

$$\lim_{n\to\infty} f_n(x) = f(x).$$

(b) Show that the convergence in (a) ~~above~~ is uniform on $[0,\infty)$.

(i.e., $f_n \to f$ uniformly on $[0,\infty)$).

Q7) For every $n \geq 1$, we define

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = \operatorname{Arctan}(nx)$$

(a) For every $x \in \mathbb{R}$, evaluate

$$\lim_{n\to\infty} f_n(x) = f(x).$$

(Find the pointwise limit of f_n on $\mathbb{R}$)

(b) Show that if $a > 0$, then f_n converges uniformly on $[a,\infty)$.

(c) Show that f_n does not converge uniformly on $(0,\infty)$.

$f, f_n : D \to \mathbb{K}^N$

$D \subseteq \mathbb{K}^d$

1) f is continuous on D
2) f is uniformly continuous on D
3) $f_n \to f$ pointwise on D
4) $f_n \to f$ uniformly on D.

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1) f is continuous at each point $x \in D$

$\forall x \in D$, f is continuous at x

$\forall x \in D$, $\forall \varepsilon > 0$, $\exists \delta = \delta(x,\varepsilon) > 0$ s.t.

$$\|f(y) - f(x)\| < \varepsilon$$

for every $y \in D$ with

$$\|x - y\| < \delta.$$

$$\left(\lim_{y \to x} f(y) = f(x) \right)$$

2) f is uniformly continuous on D if

$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

$$\|f(x) - f(y)\| < \varepsilon$$

for every $x, y \in D$ with

$$\|x - y\| < \delta.$$

3) $\forall x \in D$, $f_n(x) \to f(x)$ as $n \to \infty$

$\forall x \in D$, $\forall \varepsilon > 0$, $\exists n_{x,\varepsilon} \geq 1$ s.t.

$$\|f_n(x) - f(x)\| < \varepsilon$$

for every $n \geq n_{x,\varepsilon}$

4) $\forall \varepsilon > 0$, $\exists n_{\varepsilon} \geq 1$ s.t.

$$\|f_n(x) - f(x)\| < \varepsilon$$

for every $n \geq n_{\varepsilon}$, for every $x \in D$.

Q5) (a) $x = 0$: $f_n(0) = 0$, $\forall n \geq 1$.

$$\lim_{n \to \infty} f_n(0) = \underline{0 = f(0)}.$$

For $x > 0$, then $1 + nx \to \infty$

$$|\sin(nx)| \leq 1.$$

$$\frac{\sin(nx)}{1+nx} \to 0 \text{ by the squeeze law.}$$

$$\underline{f}(x) = \lim_{n\to\infty} f_n(x) = 0, \quad \forall x \in [0, \infty).$$

(b) Let $a > 0$.

$f_n \to f$ uniformly on $[a, \infty)$. We need to prove that

$\forall \varepsilon > 0$, $\exists n_\varepsilon \geq 1$ s.t.

$$|f_n(x) - f(x)| < \varepsilon$$

$\forall n \geq n_\varepsilon$, $\forall x \in [a, \infty)$.

Let $\varepsilon > 0$ be fixed. Evaluate

(*) $$|f_n(x) - \underbrace{f(x)}_{\text{pointwise limit}}| = \left| \frac{\sin(nx)}{1+nx} - 0 \right| = \frac{|\sin(nx)|}{1+nx} \le \underbrace{\frac{1}{1+nx} \le \frac{1}{1+na}}_{\forall x \ge a}$$

bounding $|f_n(x) - f(x)|$ by a quantity independent of x, which as a sequence in n converges to 0.

$$\left.\begin{array}{l} x \ge a > 0 \\ n \ge 1 \end{array}\right\} \Rightarrow \begin{array}{l} nx \ge na \\ 1+nx \ge 1+na \end{array}$$

Since $\frac{1}{1+na} \to 0$ as $n \to \infty$,

there exists $n_\varepsilon \ge 1$ s.t.

$$\frac{1}{1+na} < \varepsilon, \ \forall n \ge n_\varepsilon \quad (**)$$

From (*) and (**), there exists $n_\varepsilon \ge 1$ s.t.

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \ge n_\varepsilon, \ \forall x \in [a, \infty)$$

(c) $f_n \not\to f$ uniformly on $[0,\infty)$.

Assume by contradiction that $f_n \to f$ uniformly on $[0,\infty)$.

Then $\forall \varepsilon > 0$, $\exists n_\varepsilon \geq 1$ s.t.

$$\underbrace{|f_n(x) - f(x)|} < \varepsilon, \quad \forall n \geq n_\varepsilon, \quad \forall x \in [0,\infty).$$

We have (~) $\quad |f_n(x) - f(x)| = \dfrac{|\sin(nx)|}{1+nx} < \varepsilon, \quad \forall n \geq n_\varepsilon, \forall x \in [0,\infty)$

Idea: Find a sequence $x_n > 0$ s.t. $x_n \to 0$ as $n \to \infty$ (since the problem is when x approaches 0 from the right).

Such that $\dfrac{|\sin(nx_n)|}{1+nx_n} \not\to 0$ when $n \to \infty$.

$x_n = \dfrac{1}{n}$, $\quad x_n \to 0$ as $n \to \infty$.

$$\frac{|\sin(nx_n)|}{1+nx_n} = \frac{|\sin 1|}{1+1} = \frac{|\sin 1|}{2} \neq 0.$$

Hence (~) cannot hold for $x_n = \frac{1}{n}$ by choosing $\varepsilon > 0$ small. ∎

Q6) (a) $f_n(0) = 0, \quad \forall n$

$f(0) = \lim_{n\to\infty} f_n(0) = 0.$

$\forall x > 0$, we have $f_n(x) \to 0 = f(x)$.

So, $f(x) = 0 \quad \forall x \in [0,\infty)$.

(b) $f_n \to f$ uniformly on $[0,\infty)$ if and only if

$\forall \varepsilon > 0, \ \exists n_\varepsilon \geq 1$ s.t.

$|f_n(x) - 0| < \varepsilon, \ \forall n \geq n_\varepsilon, \ \forall x \in [0,\infty).$

~~Let $\varepsilon > 0$ be fixed.~~

$|f_n(x)| = x^2 e^{-nx} \leq \max_{x\in[0,\infty)} \cancel{x}^2 e^{-nx}$

$g(x) = x^2 e^{-nx}$. Find the maximum.

$g'(x) = 2x e^{-nx} - n x^2 e^{-nx} = x(2-nx)e^{-nx}$

$g'(x) = 0 \Rightarrow x = \frac{2}{n}$ critical point.

For $0 \leq x < \frac{2}{n}$, $g'(x) \geq 0$

g is increasing on $[0, \frac{2}{n})$.

if $x > \frac{2}{n}$; $g'(x) < 0$

g is decreasing on $(\frac{2}{n}, \infty)$

$x = \frac{2}{n}$ is a maximum point. Hence,

$\underline{g(x) \leq g(\frac{2}{n})} \quad \forall x \in [0,\infty)$

$g(x) \leq g(\frac{2}{n}) = \left(\frac{2}{n}\right)^2 e^{-n\cdot\frac{2}{n}}$

$= \frac{4}{n^2} e^{-2}.$

Since $\frac{4}{n^2} e^{-2} \to 0$ as $n \to \infty$, there exists

$n_\varepsilon \geq 1$ s.t $\frac{4}{n^2} e^{-2} < \varepsilon, \quad \forall n \geq n_\varepsilon.$

Hence. $\underline{|f_n(x)|} \leq g\left(\frac{2}{n}\right) = \frac{4}{n^2} e^{-2} < \underline{\varepsilon}, \quad \forall n \geq n_\varepsilon, \ \forall x \in [0, \infty).$