Review of basic topology

Idea

Introduce a class of sets suitable to describe properties of continuous functions.

This class consists of open sets. Open sets in $\mathbb{K}^N$ are constructed using open balls

$$B(x,r) := \{y \in \mathbb{K}^N : \|y - x\| < r\}.$$

Definition (open/closed sets)

- $U \subseteq \mathbb{K}^N$ is called open if $U = \emptyset$ or for every $x \in U$ there exists $r > 0$ such that $B(x,r) \subseteq U$.
- $A \subseteq \mathbb{K}^N$ is called closed if the complement $A^c = \{x \in \mathbb{K}^N : x \notin A\}$ is open.

Sometimes it is convenient to replace the base set $\mathbb{K}^N$ by a subset $D \subseteq \mathbb{K}^N$.

Definition (relatively open/closed sets)

Let $D \subseteq \mathbb{K}^N$.

- $U \subseteq D$ is called relatively open in D if $U = \emptyset$ or for every $x \in U$ there exists $r > 0$ such that $B(x,r) \cap D \subseteq U$.
- $A \subseteq D$ is called relatively closed in D if the complement $A^c \cap D$ is relatively open in D.

Note:

- For short we often say "U is open in D" or "A is closed in D"
- If $D = \mathbb{K}^N$, then open/closed and relatively open/closed are the same.
- U is relatively open in D if and only if there exists an open set V so that $U = V \cap D$ (see Question 5(b) in Tutorial 7).

Basic properties of relatively open sets

Let $D \subseteq \mathbb{K}^N$. Then

- $\emptyset$ and D are open in D;
- Arbitrary unions of relatively open sets are open in D;
- Finite intersections of relatively open sets are open in D. For a proof of these properties, see Question 5(c) in Tutorial 7.

Note:

- Many properties of continuous functions can be proved just from the above three properties.
- At a more abstract level, any collection of subsets of D with the above property is called a "topology" and a corresponding notion of continuity can be defined.
- Using de Morgan's law, the above can be rewritten in terms of closed sets.

We can look at open and closed sets associated with any set

Definition (interior, closure, boundary)

Let $A \subseteq \mathbb{K}^N$.

- $x \in A$ is an interior point of A if $B(x, r) \subseteq A$ for some $r > 0$;
- $\operatorname{int}(A) := \{x \in A : x \text{ interior point of } A\}$ interior of A;
- $\overline{A} = \{x \in \mathbb{K}^N : B(x, r) \cap A \neq \emptyset \text{ for all } r > 0\}$ closure of A;
- $\partial A = \overline{A} \setminus \operatorname{int}(A)$ boundary of A.

Major Facts:

- $\operatorname{int}(A)$ is open and $\overline{A}$ is closed;
- A open if and only if $\operatorname{int}(A) = A$;
- A closed if and only if $\overline{A} = A$;
- $x \in \overline{A}$ if and only if there exists a sequence (x_n) in A with $x_n \to x$.

Common classes of sets

Intervals.

- open intervals are open sets: (a, b), (a, ∞), $(-\infty, b)$.
- closed intervals are closed sets: $[a, b]$, $[a, \infty)$, $(-\infty, b]$.
- Intervals of the form $I = (a, b]$ are neither open nor closed and

$$\text{int}(I) = (a, b); \qquad \bar{I} = [a, b]; \qquad \partial I = \{a, b\}.$$

If $I = (a, \infty)$, then

$$\text{int}(I) = (a, \infty); \qquad \bar{I} = [a, \infty); \qquad \partial I = \{a\}.$$

Dense sets.

- $\mathbb{Q} \subseteq \mathbb{R}$. Since any interval contains a rational and an irrational number

$$\text{int}\,\mathbb{Q} = \emptyset; \qquad \overline{\mathbb{Q}} = \mathbb{R}; \qquad \partial\mathbb{Q} = \mathbb{R}.$$

Discrete sets.

- Finite sets $\{1, 2, 3\}$ or $\{a\}$ or $\{a, b, c, d\}$ are closed sets.
- Countable sets: $\{1/n \colon n \in \mathbb{N} \setminus \{0\}\}$, $\mathbb{Z}$, $\{(1 + (-1)^n/n)^n \colon n \in \mathbb{N} \setminus \{0\}\}$.
  They all have empty interior. By the sequential characterisation of the closure, their closure includes all accumulation points of the sequences. For instance

$$\overline{\{1/n \colon n \in \mathbb{N} \setminus \{0\}\}} = \{1/n \colon n \in \mathbb{N} \setminus \{0\}\} \cup \{0\}$$

  Their boundary equals their closure.

Sets defined by continuous functions.

- $f(x, y, z) = c$, that is $\{(x, y, z) \in \mathbb{R}^3 \colon f(x, y, z) = c\}$ closed sets (level surfaces)
- $f(x, y) > 3$ open set

More complicated sets. Cantor set: closed uncountable set with empty interior and no isolated points.

Characterisation of continuity using open sets

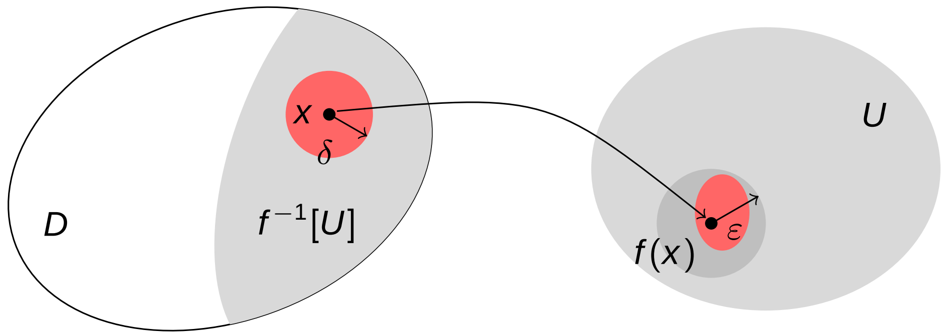
Theorem

Let $f : D \to \mathbb{K}^N$ with $D \subseteq \mathbb{K}^d$. Then the following assertions are equivalent

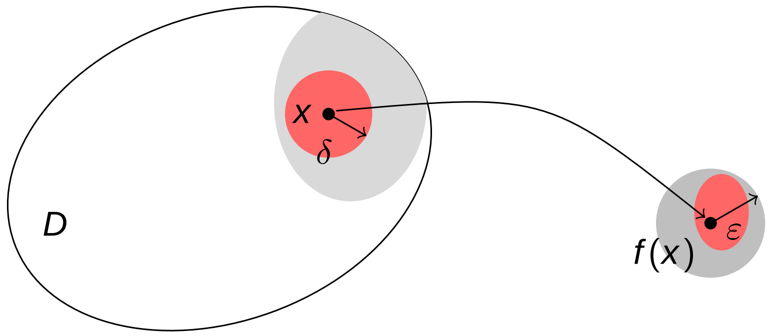
(i) f is continuous on D;

(ii) $f^{-1}[U] := \{x \in D : f(x) \in U\}$ is open in D for every $U \subseteq \mathbb{K}^N$ open.

Proof of (i) $\Rightarrow$ (ii).



Proof of (ii) $\Rightarrow$ (i). Apply (ii) to $U = B(f(x), \varepsilon)$



Consequences

- $f^{-1}[A]$ is closed in D for every $A \subseteq \mathbb{K}^N$ closed.
- In particular: Level surfaces of continuous functions are closed sets.