THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 1 (Week 2)

MATH2962: Real and Complex Analysis (Advanced) Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
Lecturer: Florica Cîrstea

Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Determine the supremum of the following sets. Determine whether it is a maximum or not.

(a) $A = \{x \in \mathbb{Q} \colon x^2 \leq 8\}$ (b) $B = \{1 - \frac{1}{n} \colon n \in \mathbb{N} \backslash \{0\}\}$ (c) $C = \{(1/2^n) \colon n \in \mathbb{N}\}$

2. Suppose A, B are non-empty subsets of $\mathbb{R}$. We set

$$A + B := \{x + y \colon x \in A, y \in B\} \quad \text{and} \quad -A := \{-x \colon x \in A\}.$$

Prove the following statements.

(a) $\sup(A) = -\inf(-A)$;

(b) $\sup(A + B) = \sup A + \sup B$;

(c) If $A \subseteq B$ then $\sup A \leq \sup B$;

*(d) If $s := \sup A \notin A$, then there exist strictly increasing $x_n \in A$ with $\sup_{n \in \mathbb{N}} x_n = s$.

3. Let $x_k > 0$ for $k = 1, \ldots, n$. Use the Cauchy Schwarz inequality to prove that

$$n^2 \leq \Big(\sum_{k=1}^{n} x_k\Big)\Big(\sum_{k=1}^{n} \frac{1}{x_k}\Big).$$

4. (a) Prove by induction that $1 + nx \leq (1 + x)^n$ for all $x \geq -1$ and $n \geq 1$ (*Bernoulli's inequality.*)

(b) Prove that $(1 + 1/n)^n \geq 2$ for all $n \in \mathbb{N}$, $n \geq 1$.

Extra questions for further practice

5. Suppose $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$ are bounded from above. Prove the following statements.

(a) $\sup(A \cup B) = \max\{\sup A, \sup B\}$;

(b) $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$. Is there always equality, or are there sets for which there is strict inequality? Prove your claim or give a counter example.

Challenge questions (optional)

6. Let $A = [a_{ij}] \in \mathbb{K}^{m \times n}$ be an $m \times n$ matrix. Define a matrix norm by

$$\|A\| := \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\Big)^{1/2}.$$

Note that this is the usual norm in $\mathbb{K}^{m \times n} = \mathbb{K}^{mn}$, and hence has all its properties.

(a) Show that $\|A\boldsymbol{x}\| \leq \|A\| \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{K}^n$.

(b) Suppose A and B are matrices such that their product AB is well defined. Show that $\|AB\| \leq \|A\| \|B\|$.