The University of Sydney
School of Mathematics and Statistics

Tutorial 1 (Week 2)

MATH2962: Real and Complex Analysis (Advanced) Semester 1, 2017

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
Lecturer: Florica C. Cîrstea

Questions marked with * are more difficult questions.

Material covered

(1) Properties of the real number system, including supremum and infimum;

(2) Inequalities, in particular the Cauchy-Schwarz inequality.

Outcomes

This tutorial helps you to

(1) be familiar with fundamental properties of real numbers;

(2) be able to write short proofs from definitions and previously established facts;

(3) work with inequalities.

Summary of essential material

Definition of a supremum and an infimum

You will need the definition of a supremum, $\sup A$, of a non-empty subset A of $\mathbb{R}$:

(i) $\sup A$ is an upper bound of A, that is, $a \leq \sup A$ for all $a \in A$;

(ii) Every upper bound of A is larger or equal to $\sup A$, that is, if m is an arbitrary upper bound of A, then $m \geq \sup A$.

As a consequence of this definition the following fact is often useful:

If m is such that $a \leq m$ for all $a \in A$, then $m \geq \sup A$.

This is a simple consequence of part (ii) of the definition, but it is applied very often. Similar statements apply to the infimum.

Some commonly used notation

We also explain some notation:

- The colon at the front or back of an equal sign:

$$X := Y \quad X \text{ is defined to be expression } Y.$$
$$X =: Y \quad Y \text{ is defined to be expression } X.$$

  The symbol near the colon is *defined* to be the expression on the other side of the equality sign. For instance $f(x) := x^2$ defines the function f at x to be x^2.

- Set notation. Given a set S we often look at subsets of elements that satisfy some conditions. We write this as

$$\{x \in S \colon \text{property satisfied by } x\}$$

Rather than a colon we sometimes use a vertical bar:

$$\{x \in X \mid \text{property satisfied by } x\}$$

Example: $\{x \in \mathbb{R} \colon x^2 < 2\}$ is the set of all real numbers with the property that $x^2 < 2$. The properties to be satisfied can be more complicated.

Questions to complete during the tutorial

1. Determine the supremum of the following sets. Determine whether it is a maximum or not. No formal proof is required.

(a) $\{x \in \mathbb{Q} \colon x^2 \leq 8\} \subseteq \mathbb{Q}$

(b) $\{1 - \frac{1}{n} \colon n \in \mathbb{N}, n \neq 0\} \subseteq \mathbb{R}$

(c) $\{\frac{1}{2^n} \colon n \in \mathbb{N}\} \subseteq \mathbb{R}$

2. Let $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$. Show that $\sup A \leq \sup B$.

3. Suppose A, B are non-empty subsets of $\mathbb{R}$. We set

$$-A := \{-x \colon x \in A\}.$$

We prove that $\sup(A) = -\inf(-A)$. To do so let $M := -\inf(-A)$.

(a) Sketch a diagram showing what is to be proved.

(b) Use the definition of a supremum to show that $M = \sup A$.

  (i) Show that M is an upper bound of A.

  (ii) Show that every upper bound of A is larger than M, that is, if m is an upper bound of A, then $m \geq M$.

4. Suppose A, B are non-empty subsets of $\mathbb{R}$. We set

$$A + B := \{x + y \colon x \in A, y \in B\}$$

We would like to show that $\sup(A + B) = \sup A + \sup B$. We do that by proving two inequalities.

(a) Using the definition of a supremum, show that $\sup(A + B) \leq \sup A + \sup B$.

(b) Show that $\sup(A + B) - y$ is an upper bound for A for all $y \in B$, and use this to show that $\sup(A + B) \geq \sup A + \sup B$.

***5.** Let $A \subseteq \mathbb{R}$ be non-empty. If $s := \sup A \notin A$, then there exist strictly increasing $x_n \in A$ with $\sup_{n \in \mathbb{N}} x_n = s$.

6. Let $x_k > 0$ for $k = 1, \ldots, n$. Use the Cauchy Schwarz inequality to prove that

$$n^2 \leq \Big(\sum_{k=1}^{n} x_k\Big)\Big(\sum_{k=1}^{n} \frac{1}{x_k}\Big).$$

7. (a) Prove by induction that $1 + nx \leq (1 + x)^n$ for all $x \geq -1$ and $n \geq 1$ (*Bernoulli's inequality.*)

(b) Prove that $(1 + 1/n)^n \geq 2$ for all $n \in \mathbb{N}$, $n \geq 1$.

Extra questions for further practice

8. Suppose $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$ are bounded from above. Prove the following statements.

(a) $\sup(A \cup B) = \max\{\sup A, \sup B\}$;

(b) $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$. Is there always equality, or are there sets for which there is strict inequality? Prove your claim or give a counter example.

Challenge questions (optional)

9. Let $A = [a_{ij}] \in \mathbb{K}^{m \times n}$ be an $m \times n$ matrix. Define a matrix norm by

$$\|A\| := \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

Note that this is the usual norm in $\mathbb{K}^{m \times n} = \mathbb{K}^{mn}$, and hence has all its properties.

(a) Show that $\|A\boldsymbol{x}\| \leq \|A\|\|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{K}^n$.

(b) Suppose A and B are matrices such that their product AB is well defined. Show that $\|AB\| \leq \|A\|\|B\|$.