Questions marked with * are more difficult questions.

Material covered

(1) Properties of the real number system, including supremum and infimum;
(2) Inequalities, in particular the Cauchy-Schwarz inequality.

Outcomes

This tutorial helps you to

(1) be familiar with fundamental properties of real numbers;
(2) be able to write short proofs from definitions and previously established facts;
(3) work with inequalities.

Summary of essential material

Definition of a supremum and an infimum

You will need the definition of a supremum, sup $A$, of a non-empty subset $A$ of $\mathbb{R}$:

(i) sup $A$ is an upper bound of $A$, that is, $a \leq \text{sup } A$ for all $a \in A$;

(ii) Every upper bound of $A$ is larger or equal to sup $A$, that is, if $m$ is an arbitrary upper bound of $A$, then $m \geq \text{sup } A$.

As a consequence of this definition the following fact is often useful:

\[ \text{If } m \text{ is such that } a \leq m \text{ for all } a \in A, \text{ then } m \geq \text{sup } A. \]

This is a simple consequence of part (ii) of the definition, but it is applied very often. Similar statements apply to the infimum.

Some commonly used notation

We also explain some notation:

- The colon at the front or back of an equal sign:

\[ X := Y \quad X \text{ is defined to be expression } Y. \]
\[ X =: Y \quad Y \text{ is defined to be expression } X. \]

The symbol near the colon is defined to be the expression on the other side of the equality sign. For instance $f(x) := x^2$ defines the function $f$ at $x$ to be $x^2$. 
Set notation. Given a set $S$ we often look at subsets of elements that satisfy some conditions. We write this as

$$\{ x \in S : \text{property satisfied by } x \}$$

Rather than a colon we sometimes use a vertical bar:

$$\{ x \in X | \text{property satisfied by } x \}$$

Example: $\{ x \in \mathbb{R} : x^2 < 2 \}$ is the set of all real numbers with the property that $x^2 < 2$. The properties to be satisfied can be more complicated.

Questions to complete during the tutorial

1. Determine the supremum of the following sets. Determine whether it is a maximum or not. No formal proof is required.
   (a) $\{ x \in \mathbb{Q} : x^2 \leq 8 \} \subseteq \mathbb{Q}$
   (b) $\{ 1 - \frac{1}{n} : n \in \mathbb{N}, n \neq 0 \} \subseteq \mathbb{R}$
   (c) $\{ \frac{1}{2n} : n \in \mathbb{N} \} \subseteq \mathbb{R}$

2. Let $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$. Show that $\text{sup } A \leq \text{sup } B$.

3. Suppose $A, B$ are non-empty subsets of $\mathbb{R}$. We set

$$-A := \{ -x : x \in A \}.$$  

We prove that $\text{sup}(A) = -\text{inf}(-A)$. To do so let $M := -\text{inf}(-A)$.
   (a) Sketch a diagram showing what is to be proved.
   (b) Use the definition of a supremum to show that $M = \text{sup } A$.
      (i) Show that $M$ is an upper bound of $A$.
      (ii) Show that every upper bound of $A$ is larger than $M$, that is, if $m$ is an upper bound of $A$, then $m \geq M$.

4. Suppose $A, B$ are non-empty subsets of $\mathbb{R}$. We set

$$A + B := \{ x + y : x \in A, y \in B \}$$

We would like to show that $\text{sup}(A + B) = \text{sup } A + \text{sup } B$. We do that by proving two inequalities.
   (a) Using the definition of a supremum, show that $\text{sup}(A + B) \leq \text{sup } A + \text{sup } B$.
   (b) Show that $\text{sup}(A + B) - y$ is an upper bound for $A$ for all $y \in B$, and use this to show that $\text{sup}(A + B) \geq \text{sup } A + \text{sup } B$.

*5. Let $A \subseteq \mathbb{R}$ be non-empty. If $s := \text{sup } A \notin A$, then there exist strictly increasing $x_n \in A$ with $\text{sup } x_n = s$.

6. Let $x_k > 0$ for $k = 1, \ldots, n$. Use the Cauchy Schwarz inequality to prove that

$$n^2 \leq \left( \sum_{k=1}^{n} x_k \right) \left( \sum_{k=1}^{n} \frac{1}{x_k} \right).$$
7. (a) Prove by induction that \(1 + nx \leq (1 + x)^n\) for all \(x \geq -1\) and \(n \geq 1\) (Bernoulli’s inequality.)

(b) Prove that \((1 + 1/n)^n \geq 2\) for all \(n \in \mathbb{N}, n \geq 1\).

Extra questions for further practice

8. Suppose \(A, B \subseteq \mathbb{R}, A, B \neq \emptyset\) are bounded from above. Prove the following statements.

   (a) \(\sup(A \cup B) = \max\{\sup A, \sup B\}\);

   (b) \(\sup(A \cap B) \leq \min\{\sup A, \sup B\}\). Is there always equality, or are there sets for which there is strict inequality? Prove your claim or give a counter example.

Challenge questions (optional)

9. Let \(A = [a_{ij}] \in \mathbb{K}^{m \times n}\) be an \(m \times n\) matrix. Define a matrix norm by

\[
\|A\| := \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.
\]

Note that this is the usual norm in \(\mathbb{K}^{m\times n} = \mathbb{K}^{mn}\), and hence has all its properties.

(a) Show that \(\|Ax\| \leq \|A\|\|x\|\) for all \(x \in \mathbb{K}^n\).

(b) Suppose \(A\) and \(B\) are matrices such that their product \(AB\) is well defined. Show that \(\|AB\| \leq \|A\|\|B\|\).