The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 1 (Week 2)

MATH2962: Real and Complex Analysis (Advanced) Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Determine the supremum of the following sets. Determine whether it is a maximum or not.

(a) $A = \{x \in \mathbb{Q} \colon x^2 \leq 8\}$

Solution: $\sup A = 2\sqrt{2}$ is not a maximum because $2\sqrt{2} \notin \mathbb{Q}$.

(b) $B = \{1 - \frac{1}{n} \colon n \in \mathbb{N} \setminus \{0\}\}$

Solution: $\sup B = 1$ is not a maximum as $1 - 1/n < 1$ for all $n \in \mathbb{N} \setminus \{0\}$

(c) $C = \{(1/2^n) \colon n \in \mathbb{N}\}$

Solution: $\sup C = 1 = \max C$ since $(1/2)^0 = 1$ is an element of C.

2. Suppose A, B are non-empty subsets of $\mathbb{R}$. We set

$$A + B := \{x + y \colon x \in A, y \in B\} \quad \text{and} \quad -A := \{-x \colon x \in A\}.$$

Prove the following statements.

(a) $\sup(A) = -\inf(-A)$;

Solution: Set $M := -\inf(-A)$. We want to show that $\sup A = M$. We first show that M is an upper bound for A. Since $M = -\inf(-A)$ we have $-M \leq -x$ for all $x \in A$ and so $x \leq M$ for all $x \in A$. Hence M is an upper bound for A. Next we need to show that given an arbitrary upper bound m of A, it follows that $M \leq m$. Since m is an upper bound for A we have $x \leq m$ and so $-m \leq -x$ for all $x \in A$. This implies that $-m$ is a lower bound for $-A$. Hence $-m \leq -M$ or $M \leq m$ as required.

(b) $\sup(A + B) = \sup A + \sup B$;

Solution: The idea is to prove two inequalities: $\sup(A + B) \leq \sup A + \sup B$, and $\sup(A+B) \geq \sup A + \sup B$. Since $\sup A$ is an upper bound for A we have $x \leq \sup A$ for all $x \in A$. Similarly, $y \leq \sup B$ for all $y \in B$. Hence $x + y \leq \sup A + \sup B$ for all $x \in A$ and $y \in B$, that is, $\sup A + \sup B$ is an upper bound for $A + B$. Hence by definition of a supremum $\sup(A + B) \leq \sup A + \sup B$.

We next show the opposite inequality. By definition of $A + B$ and $\sup(A + B)$ we have $x = (x + y) - y \leq \sup(A + B) - y$ for all $x \in A$ and $y \in B$. Hence if we fix $y \in B$, then $\sup(A + B) - y$ is an upper bound for A and so by definition of $\sup A$ we have $\sup A \leq \sup(A + B) - y$ for every $y \in B$. Rearranging we get $y \leq \sup(A + B) - \sup A$ for all $y \in B$, so again by the definition of a supremum $\sup B \leq \sup(A + B) - \sup A$. Hence $\sup A + \sup B \leq \sup(A + B)$. Combining the two inequalities we get equality.

(c) If $A \subseteq B$ then $\sup A \leq \sup B$;

Solution: Suppose that m is an upper bound of B. Then $x \leq m$ for all $x \in A$ as $A \subseteq B$. This means that every upper bound of B is also an upper bound of A. In particular, $\sup B$ is an upper bound for A. By definition of the supremum, $\sup A \leq \sup B$.

*(d) If $s := \sup A \notin A$, then there exist strictly increasing $x_n \in A$ with $\sup_{n\in\mathbb{N}} x_n = s$.

Solution: By what we proved in lectures (Lemma 2.1 in the notes) there exists $x_0 \in A$ such that $s - 1 < x_0 \leq s$. As $\sup A \notin A$ we must have $x_0 < \sup A$. Hence we apply the same fact again to find $x_1 \in A$ such that $\max\{x_0, s - 1/2\} < x_1 < s$. The strict inequality holds since $\sup A \notin A$. Hence we can inductively construct $x_n \in A$ such that $\max\{x_{n-1}, s - 2^{-n}\} < x_n < s$. By construction (x_n) is strictly increasing and $\sup_{n\in\mathbb{N}} x_n = s$.

3. Let $x_k > 0$ for $k = 1, \dots, n$. Use the Cauchy Schwarz inequality to prove that

$$n^2 \leq \Big(\sum_{k=1}^{n} x_k\Big)\Big(\sum_{k=1}^{n} \frac{1}{x_k}\Big).$$

Solution: By the Cauchy Schwarz inequalaity

$$n = \sum_{k=1}^{n} 1 = \sum_{k=1}^{n} \sqrt{x_k}\sqrt{\frac{1}{x^k}} \leq \Big(\sum_{k=1}^{n} x_k\Big)^{1/2}\Big(\sum_{k=1}^{n} \frac{1}{x_k}\Big)^{1/2}.$$

Now the required inequality follows by taking the square of the above inequality.

4. (a) Prove by induction that $1+nx \leq (1+x)^n$ for all $x \geq -1$ and $n \geq 1$ (*Bernoulli's inequality.*)

Solution: If $n = 1$, then the inequality is obvious since $(1 + x)^1 = 1 + 1x$. Now we do the induction step. The induction hypothesis is that the inequality is true for some $n \geq 1$. We claim it is also true for $n + 1$. More explicitly, we assume that $n \geq 1$ and $(1+x)^n \geq 1+nx$ if $x \geq -1$. We need to prove that $(1+x)^{n+1} \geq 1+(n+1)x$. From that assumption we get

$$(1 + x)^{n+1} = (1 + x)^n(1 + x) \geq (1 + nx)(1 + x) = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x$$

with the last inequality being strict if $x \neq 0$.

(b) Prove that $(1 + 1/n)^n \geq 2$ for all $n \in \mathbb{N}$, $n \geq 1$.

Solution: By Bernoulli's inequality from (a) we get

$$\Big(1 + \frac{1}{n}\Big)^n \geq 1 + n\frac{1}{n} = 2$$

for all $n \geq 1$.

Extra questions for further practice

5. Suppose $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$ are bounded from above. Prove the following statements.

(a) $\sup(A \cup B) = \max\{\sup A, \sup B\}$;

Solution: Clearly $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Hence $\sup A \leq \sup(A \cup B)$ and $\sup B \leq \sup(A \cup B)$ (by Question **2**(c)) and thus

$$\max\{\sup A, \sup B\} \leq \sup(A \cup B). \tag{1}$$

If $x \in A \cup B$, then $x \in A$ or $x \in B$ and thus $x \leq \sup A$ or $x \leq \sup B$. Hence, $x \leq \max\{\sup A, \sup B\}$ for all $x \in A \cup B$. By definition of a supremum

$$\sup(A \cup B) \leq \max\{\sup A, \sup B\}. \tag{2}$$

Combining (1) and (2) the claim follows.

(b) $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$. Is there always equality, or are there sets for which there is strict inequality? Prove your claim or give a counter example.

Solution: If $A \cap B = \emptyset$, then by definition $\sup(A \cap B) = -\infty$ and obviously $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$. Suppose now that $A \cap B \neq \emptyset$. Then if $x \in A \cap B$, then in particular $x \in A$ and $x \in B$. Hence $x \leq \sup A$ and $x \leq \sup B$. Hence

$$\sup(A \cap B) \leq \min\{\sup A, \sup B\}.$$

In general there is no equality. As a counter example set $A = [0, 2] \cup [4, 5]$ and $B := [2, 3]$. Then $\sup A = 5$ and $\sup B = 3$, so $\min\{\sup A, \sup B\} = 3$. On the other hand, $A \cap B = \{2\}$, so $\sup(A \cap B) = 2$. Hence $2 = \sup(A \cap B) < \min\{\sup A, \sup B\} = 3$. There are many other possible examples.

Challenge questions (optional)

6. Let $A = [a_{ij}] \in \mathbb{K}^{m \times n}$ be an $m \times n$ matrix. Define a matrix norm by

$$\|A\| := \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\Big)^{1/2}.$$

Note that this is the usual norm in $\mathbb{K}^{m \times n} = \mathbb{K}^{mn}$, and hence has all its properties.

(a) Show that $\|A\boldsymbol{x}\| \leq \|A\| \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{K}^n$.

Solution: Denote the row vectors of A by $\boldsymbol{r}_1, \dots \boldsymbol{r}_m$. Then by definition of $A\boldsymbol{x}$ we have

$$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{x} \\ \vdots \\ \boldsymbol{r}_m \cdot \boldsymbol{x} \end{bmatrix}$$

Now by definition of the norm and the Cauchy-Schwarz inequality

$$\|A\boldsymbol{x}\|^2 = \sum_{i=1}^{m} |\boldsymbol{r}_i \cdot \bar{\boldsymbol{x}}|^2 \leq \sum_{i=1}^{m} \|\boldsymbol{r}_i\|^2 \|\boldsymbol{x}\|^2 = \|\boldsymbol{x}\|^2 \sum_{i=1}^{m} \|\boldsymbol{r}_i\|^2 = \|A\|^2 \|\boldsymbol{x}\|^2.$$

The complex conjugate $\bar{\boldsymbol{x}}$ appears because of the definition of the inner product in case $\mathbb{K} = \mathbb{C}$. Taking square roots we get the required inequality.

(b) Suppose A and B are matrices such that their product AB is well defined. Show that $\|AB\| \leq \|A\| \|B\|$.

Solution: If A has n columns, then AB is defined if B has n rows. Suppose that $\boldsymbol{r}_1, \dots \boldsymbol{r}_m$ are the row vectors of A and $\boldsymbol{c}_1, \dots \boldsymbol{c}_d$ are its column vectors of B. Then by definition of matrix multiplication

$$AB = \begin{bmatrix} \boldsymbol{r}_1 \cdot \bar{\boldsymbol{c}}_1 & \dots & \boldsymbol{r}_1 \cdot \bar{\boldsymbol{c}}_d \\ \vdots & \ddots & \vdots \\ \boldsymbol{r}_m \cdot \bar{\boldsymbol{c}}_1 & \dots & \boldsymbol{r}_m \cdot \bar{\boldsymbol{c}}_d \end{bmatrix}.$$

Now by definition of the norm and the Cauchy-Schwarz inequality

$$\begin{aligned} \|AB\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{d} |\boldsymbol{r}_i \cdot \bar{\boldsymbol{c}}_j|^2 \leq \sum_{i=1}^{m} \sum_{j=1}^{d} \|\boldsymbol{r}_i\|^2 \|\boldsymbol{c}_j\|^2 \\ = \Big(\sum_{i=1}^{m} \|\boldsymbol{r}_i\|^2\Big) \Big(\sum_{j=1}^{d} \|\boldsymbol{c}_j\|^2\Big) = \|A\|^2 \|B\|^2. \end{aligned}$$

Taking square roots we get the required inequality.