Material covered

(1) Properties of the real number system, including supremum and infimum;
(2) Inequalities, in particular the Cauchy-Schwarz inequality.

Outcomes

This tutorial helps you to

(1) be familiar with fundamental properties of real numbers;
(2) be able to write short proofs from definitions and previously established facts;
(3) work with inequalities.

Summary of essential material

Definition of a supremum and an infimum

You will need the definition of a supremum, sup $A$, of a non-empty subset $A$ of $\mathbb{R}$:

(i) sup $A$ is an upper bound of $A$, that is, $a \leq \text{sup} A$ for all $a \in A$;

(ii) Every upper bound of $A$ is larger or equal to sup $A$, that is, if $m$ is an arbitrary upper bound of $A$, then $m \geq \text{sup} A$.

As a consequence of this definition the following fact is often useful:

$\text{If } m \text{ is such that } a \leq m \text{ for all } a \in A, \text{ then } m \geq \text{sup} A.$

This is a simple consequence of part (ii) of the definition, but it is applied very often. Similar statements apply to the infimum.

Some commonly used notation

We also explain some notation:

- The colon at the front or back of an equal sign:
  
  $X := Y$  \hspace{1cm} $X$ is defined to be expression $Y$.
  $X =: Y$  \hspace{1cm} $Y$ is defined to be expression $X$.

  The symbol near the colon is \textit{defined} to be the expression on the other side of the equality sign. For instance $f(x) := x^2$ defines the function $f$ at $x$ to be $x^2$. 
• Set notation. Given a set $S$ we often look at subsets of elements that satisfy some conditions. We write this as

$$\{x \in S : \text{property satisfied by } x\}$$

Rather than a colon we sometimes use a vertical bar:

$$\{x \in X | \text{property satisfied by } x\}$$

Example: $\{x \in \mathbb{R} : x^2 < 2\}$ is the set of all real numbers with the property that $x^2 < 2$. The properties to be satisfied can be more complicated.

Questions to complete during the tutorial

1. Determine the supremum of the following sets. Determine whether it is a maximum or not. No formal proof is required.
   (a) $\{x \in \mathbb{Q} : x^2 \leq 8\} \subseteq \mathbb{Q}$
   **Solution:** The supremum is $2\sqrt{2}$. It is not a maximum because $2\sqrt{2} \notin \mathbb{Q}$.
   (b) $\{1 - \frac{1}{n} : n \in \mathbb{N}, n \neq 0\} \subseteq \mathbb{R}$
   **Solution:** The supremum is 1. It is not a maximum as $1 - 1/n < 1$ for all $n \in \mathbb{N}$
   (c) $\{\frac{1}{2n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$
   **Solution:** The supremum is 1. It is a maximum since $(1/2)^0 = 1$ is an element of the given set.

2. Let $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$. Show that $\sup A \leq \sup B$.
   **Solution:** Suppose that $m$ is an upper bound of $B$. Then $x \leq m$ for all $x \in A$ as $A \subseteq B$. This means that every upper bound of $B$ is also an upper bound of $A$. In particular, $\sup B$ is an upper bound for $A$. By definition of the supremum, $\sup A \leq \sup B$.

3. Suppose $A, B$ are non-empty subsets of $\mathbb{R}$. We set

$$-A := \{-x : x \in A\}.$$

We prove that $\sup(A) = -\inf(-A)$. To do so let $M := -\inf(-A)$.

(a) Sketch a diagram showing what is to be proved.
   **Solution:** The set $-A$ is the reflection of $A$ at the origin 0.

(b) Use the definition of a supremum to show that $M = \sup A$.
   (i) Show that $M$ is an upper bound of $A$.
   **Solution:** Since $M = -\inf(-A)$ we have $-M \leq -x$ for all $x \in A$ and so $x \leq M$ for all $x \in A$. Hence $M$ is an upper bound for $A$.
   (ii) Show that every upper bound of $A$ is larger than $M$, that is, if $m$ is an upper bound of $A$, then $m \geq M$.
   **Solution:** Let $m$ be an arbitrary upper bound of $A$. Hence, $x \leq m$ and so $-m \leq -x$ for all $x \in A$. This implies that $-m$ is a lower bound for $-A$. Hence $-m \leq -M$ or $M \leq m$ as required.
4. Suppose $A, B$ are non-empty subsets of $\mathbb{R}$. We set

$$A + B := \{x + y : x \in A, y \in B\}$$

We would like to show that $\sup(A + B) = \sup A + \sup B$. We do that by proving two inequalities.

(a) Using the definition of a supremum, show that $\sup(A + B) \leq \sup A + \sup B$.

Solution: Since $\sup A$ is an upper bound for $A$ we have $x \leq \sup A$ for all $x \in A$. Similarly, $y \leq \sup B$ for all $y \in B$. Hence $x + y \leq \sup A + \sup B$ for all $x \in A$ and $y \in B$, that is, $\sup A + \sup B$ is an upper bound for $A + B$. Hence by definition of a supremum $\sup(A + B) \leq \sup A + \sup B$.

(b) Show that $\sup(A + B) - y$ is an upper bound for $A$ for all $y \in B$, and use this to show that $\sup(A + B) \geq \sup A + \sup B$.

Solution: By definition of $A + B$ and $\sup(A + B)$ we have $x = (x + y) - y \leq \sup(A + B) - y$ for all $x \in A$ and $y \in B$. Hence if we fix $y \in B$, then $\sup(A + B) - y$ is an upper bound for $A$ and so by definition of $\sup A$ we have $\sup A \leq \sup(A + B) - y$ for every $y \in B$. Rearranging we get $y \leq \sup(A + B) - \sup A$ for all $y \in B$, so again by the definition of a supremum $\sup B \leq \sup(A + B) - \sup A$. Hence $\sup A + \sup B \leq \sup(A + B)$. Combining the two inequalities we get equality.

*5. Let $A \subseteq \mathbb{R}$ be non-empty. If $s := \sup A \not\in A$, then there exist strictly increasing $x_n \in A$ with $\sup_{n \in \mathbb{N}} x_n = s$.

Solution: First assume that $A$ is bounded from above, so that $s < \infty$. By what we proved in lectures (Lemma 2.1 in the notes) there exists $x_0 \in A$ such that $s - 1 < x_0 \leq s$. As $\sup A \not\in A$ we must have $x_0 < \sup A$. Hence we apply the same fact again to find $x_1 \in A$ such that $\max\{x_0, s - 1/2\} < x_1 < s$. The strict inequality holds since $\sup A \not\in A$. Hence we can inductively construct $x_n \in A$ such that $\max\{x_{n-1}, s - 2^{-n}\} < x_n < s$. By construction $(x_n)$ is strictly increasing and $\sup_{n \in \mathbb{N}} x_n = s$.

We can give a similar argument if $A$ is not bounded from above, that is, $s = \infty$. As $A$ is non-empty we can pick $x_0 \in A$. Since $\max\{x_0, 1\}$ is not an upper bound of $A$, there exist $x_1 \in A$ such that $x_1 > \max\{x_0, 1\}$. Proceeding inductively, we construct $x_n \in A$ such that $\max\{n, x_{n-1}\} < x_n$ for all $n \geq 1$. Hence, $x_n$ is a strictly increasing sequence in $A$ with $x_n \to s = \infty$ as $n \to \infty$.

6. Let $x_k > 0$ for $k = 1, \ldots, n$. Use the Cauchy Schwarz inequality to prove that

$$n^2 \leq \left( \sum_{k=1}^{n} x_k \right) \left( \sum_{k=1}^{n} \frac{1}{x_k} \right).$$

Solution: By the Cauchy Schwarz inequality

$$n = \sum_{k=1}^{n} 1 = \sum_{k=1}^{n} \sqrt{x_k} \sqrt{\frac{1}{x_k}} \leq \left( \sum_{k=1}^{n} x_k \right)^{1/2} \left( \sum_{k=1}^{n} \frac{1}{x_k} \right)^{1/2}.$$

Now the required inequality follows by taking the square of the above inequality.

7. (a) Prove by induction that $1 + nx \leq (1 + x)^n$ for all $x \geq -1$ and $n \geq 1$ (Bernoulli’s inequality.)
Solution: If \( n = 1 \), then the inequality is obvious since \((1 + x)^1 = 1 + 1x\). Now we do the induction step. The induction hypothesis is that the inequality is true for some \( n \geq 1 \). We claim it is also true for \( n + 1 \). More explicitly, we assume that \( n \geq 1 \) and \((1 + x)^n \geq 1 + nx\) if \( x \geq -1 \). We need to prove that \((1 + x)^{n+1} \geq 1 + (n + 1)x\). From that assumption we get
\[
(1 + x)^n (1 + x) \geq (1 + nx)(1 + x) = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x
\]
with the last inequality being strict if \( x \neq 0 \).

(b) Prove that \((1 + 1/n)^n \geq 2\) for all \( n \in \mathbb{N}, n \geq 1 \).

Solution: By Bernoulli’s inequality from (a) we get
\[
\left(1 + \frac{1}{n}\right)^n \geq 1 + \frac{1}{n} = 2
\]
for all \( n \geq 1 \).

Extra questions for further practice

8. Suppose \( A, B \subseteq \mathbb{R}, A, B \neq \emptyset \) are bounded from above. Prove the following statements.

(a) \( \sup(A \cup B) = \max\{\sup A, \sup B\} \);

Solution: Clearly \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \). Hence \( \sup A \leq \sup(A \cup B) \) and \( \sup B \leq \sup(A \cup B) \) (by Question 4(2)) and thus
\[
\max\{\sup A, \sup B\} \leq \sup(A \cup B).
\]
If \( x \in A \cup B \), then \( x \in A \) or \( x \in B \) and thus \( x \leq \sup A \) or \( x \leq \sup B \). Hence, \( x \leq \max\{\sup A, \sup B\} \) for all \( x \in A \cup B \). By definition of a supremum
\[
\sup(A \cup B) \leq \max\{\sup A, \sup B\}.
\]
Combining (1) and (2) the claim follows.

(b) \( \sup(A \cap B) \leq \min\{\sup A, \sup B\} \). Is there always equality, or are there sets for which there is strict inequality? Prove your claim or give a counter example.

Solution: If \( A \cap B = \emptyset \), then by definition \( \sup(A \cap B) = -\infty \) and obviously \( \sup(A \cap B) \leq \min\{\sup A, \sup B\} \). Suppose now that \( A \cap B \neq \emptyset \). Then if \( x \in A \cap B \), then in particular \( x \in A \) and \( x \in B \). Hence \( x \leq \sup A \) and \( x \leq \sup B \). Hence
\[
\sup(A \cap B) \leq \min\{\sup A, \sup B\}.
\]
In general there is no equality. As a counter example set \( A = [0, 2] \cup [4,5] \) and \( B := [2,3] \). Then \( \sup A = 5 \) and \( \sup B = 3 \), so \( \min\{\sup A, \sup B\} = 3 \). On the other hand, \( A \cap B = \{2\} \), so \( \sup(A \cap B) = 2 \). Hence \( 2 = \sup(A \cap B) < \min\{\sup A, \sup B\} = 3 \). There are many other possible examples.

Challenge questions (optional)

9. Let \( A = [a_{ij}] \in \mathbb{K}^{m \times n} \) be an \( m \times n \) matrix. Define a matrix norm by
\[
\|A\| := \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}.
\]
Note that this is the usual norm in \( \mathbb{K}^{m \times n} = \mathbb{K}^{mn} \), and hence has all its properties.
(a) Show that $\|Ax\| \leq \|A\|\|x\|$ for all $x \in \mathbb{K}^n$.

**Solution:** Denote the row vectors of $A$ by $r_1, \ldots, r_m$. Then by definition of $Ax$ we have

$$Ax = \begin{bmatrix} r_1 \cdot \bar{x} \\ \vdots \\ r_m \cdot \bar{x} \end{bmatrix}$$

Now by definition of the norm and the Cauchy-Schwarz inequality

$$\|Ax\|^2 = \sum_{i=1}^{m} |r_i \cdot \bar{x}|^2 \leq \sum_{i=1}^{m} \|r_i\|^2 \|x\|^2 = \|x\|^2 \sum_{i=1}^{m} |r_i|^2 = \|A\|^2 \|x\|^2.$$  

The complex conjugate $\bar{x}$ appears because of the definition of the inner product in case $\mathbb{K} = \mathbb{C}$. Taking square roots we get the required inequality.

(b) Suppose $A$ and $B$ are matrices such that their product $AB$ is well defined. Show that $\|AB\| \leq \|A\|\|B\|$.

**Solution:** If $A$ has $n$ columns, then $AB$ is defined if $B$ has $n$ rows. Suppose that $r_1, \ldots, r_m$ are the row vectors of $A$ and $c_1, \ldots, c_d$ are its column vectors of $B$. Then by definition of matrix multiplication

$$AB = \begin{bmatrix} r_1 \cdot \bar{c}_1 & \ldots & r_1 \cdot \bar{c}_d \\ \vdots & \ddots & \vdots \\ r_m \cdot \bar{c}_1 & \ldots & r_m \cdot \bar{c}_d \end{bmatrix}.$$  

Now by definition of the norm and the Cauchy-Schwarz inequality

$$\|AB\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{d} |r_i \cdot \bar{c}_j|^2 \leq \sum_{i=1}^{m} \sum_{j=1}^{d} \|r_i\|^2 \|c_j\|^2 = \left(\sum_{i=1}^{m} \|r_i\|^2\right) \left(\sum_{j=1}^{d} \|c_j\|^2\right) = \|A\|^2 \|B\|^2.$$  

Taking square roots we get the required inequality.