

Tutorial 2 (Week 3)

MATH2962: Real and Complex Analysis (Advanced)

Semester 1, 2012

Web Page: <http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/>

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Questions to complete during the tutorial

1. Show that the following sequences are monotone. Also check whether they are bounded and therefore convergent. If possible determine the limit.

(a) $x_n = \frac{n}{e} + e^{-n}$;

(c) $u_0 > 0, u_{n+1} = \frac{u_n}{\sqrt{1+u_n^2}}, n \in \mathbb{N}$;

(b) $s_n = \sum_{k=1}^n \frac{1}{k}$;

(d) $v_0 = 0, v_{n+1} = \sqrt{3v_n + 4}, n \in \mathbb{N}$.

2. Determine the limit of the following sequences (x_n) as $n \rightarrow \infty$ if it exists.

(a) $x_n = \frac{2n^2 + 2^n}{n^{100} + 3 \cdot 2^n}$;

(c) $x_n = \frac{1 + 2 + \dots + n}{n + 2} - \frac{n}{2}$;

(b) $x_n = (n - \sqrt{n(n-1)})$;

(d) $x_n = (a^n + b^n)^{1/n}$ if $0 \leq a \leq b$;

3. (a) Let $a > 0$. Define a sequence recursively by choosing $x_0 > 0$ arbitrary and setting $x_{n+1} := \frac{1}{2} \left(x_n + \frac{a}{x_n} \right)$ for all $n \in \mathbb{N}$. Prove that $(x_n)_{n \geq 1}$ is decreasing and that $x_n \rightarrow \sqrt{a}$ as $n \rightarrow \infty$. (This is a practical numerical method to compute square roots as (x_n) converges very fast.)

- (b) Set $a = x_0 = 2$ and use (a) to compute x_3 . Compare the result to $\sqrt{2}$ from your calculator.

Extra questions for further practice

4. Let $a \in \mathbb{C}$ with $a \neq 1$. Use induction by n to prove that

$$s_n := \sum_{k=0}^n a^k = \frac{1 - a^{n+1}}{1 - a}$$

for all $n \in \mathbb{N}$. Conclude that (s_n) converges if and only if $|a| < 1$. In case of convergence show that $s_n \rightarrow (1 - a)^{-1}$ as $n \rightarrow \infty$.

Challenge questions (optional)

- *5. Consider the sequence given by $s_n := \sum_{k=0}^n \frac{1}{k!}$. In lectures it is shown that $s_n \rightarrow e$.

- (a) Prove that $s_n < e < s_n + \frac{1}{n!n}$ for all $n \geq 1$. Determine e to four decimal places.
(b) Use (a) to show that e is irrational. (Give a proof by contradiction.)