THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial 2 (Week 3)

MATH2962: Real and Complex Analysis (Advanced) — Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions to complete during the tutorial

1. Show that the following sequences are monotone. Also check whether they are bounded and therefore convergent. If possible determine the limit.

(a) $x_n = \dfrac{n}{e} + e^{-n}$;

Solution: Consider

$$x_{n+1} - x_n = \frac{n+1}{e} + e^{-(n+1)} - \frac{n}{e} - e^{-n} = \frac{1}{e} + e^{-n}\Big(\frac{1}{e} - 1\Big) > \frac{1}{e} - \frac{1}{e^n} > 0$$

for all $n \in \mathbb{N}$, so (x_n) is strictly increasing. Because of the term n/e the sequence is clearly unbounded.

(b) $s_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$;

Solution: The sequence is clearly increasing. Note that $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$ by the formula for geometric progression. Hence, by grouping the terms into 1, then 2, 4, …, 2^{n-1} we have

$$s_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k} = 1 + \frac{1}{2} + \underbrace{\Big(\frac{1}{3} + \frac{1}{4}\Big)}_{>2\cdot\frac{1}{4}=\frac{1}{2}} + \underbrace{\Big(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\Big)}_{>4\cdot\frac{1}{8}=\frac{1}{2}} + \cdots + \underbrace{\Big(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\Big)}_{>2^{n-1}\cdot\frac{1}{2^n}=\frac{1}{2}}$$
$$> 1 + \underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{n \text{ terms}} > \frac{n}{2}$$

for all $n \in \mathbb{N}$. Hence (s_n) is unbounded and so $s_n \to \infty$.

(c) $u_0 > 0$, $u_{n+1} = \dfrac{u_n}{\sqrt{1+u_n^2}}$, $n \in \mathbb{N}$;

Solution: We show that u_n is strictly decreasing. Clearly $u_n > 0$ and so

$$0 < u_{n+1} = \frac{u_n}{\sqrt{1+u_n^2}} < u_n,$$

so (u_n) is decreasing. Hence (u_n) is bounded and therefore has a limit u. We can determine u by letting $n \to \infty$:

$$\begin{array}{ccc} u_{n+1} & = & \dfrac{u_n}{\sqrt{1+u_n^2}} \\ \downarrow & & \downarrow \\ u & = & \dfrac{u}{\sqrt{1+u^2}}. \end{array}$$

Solving for u we get $u(\sqrt{1+u^2} - 1) = 0$, so $u = 0$.

(d) $v_0 = 0$, $v_{n+1} = \sqrt{3v_n + 4}$, $n \in \mathbb{N}$.

Solution: We show by induction that (v_n) is strictly increasing. Clearly $v_1 = \sqrt{4} = 2 > 0 = v_0$. Suppose now that $v_{n-1} < v_n$. Then

$$v_{n+1} = \sqrt{3v_n + 4} > \sqrt{3v_{n-1} + 4} = v_n,$$

so (v_n) is increasing. We now show that (v_n) is bounded by 4 by induction. By assumption $v_0 = 0 < 4$. If $v_n < 4$, then

$$v_{n+1} = \sqrt{3v_n + 4} < \sqrt{3 \cdot 4 + 4} = \sqrt{16} = 4,$$

so the claim follows. Hence (v_n) has a limit v. We can determine u by letting $n \to \infty$:

$$\begin{array}{ccc} v_{n+1} & = & \sqrt{3v_n + 4} \\ \Big\downarrow & & \downarrow \\ v & = & \sqrt{3v + 4} \end{array}$$

Solving for v we get $v^2 = 3v + 4$. Solving the quadratic yields $v = -1$ and $v = 4$. Since the sequence is positive and increasing the only possibility is that the limit is $v = 4$.

2. Determine the limit of the following sequences (x_n) as $n \to \infty$ if it exists.

(a) $x_n = \dfrac{2n^2 + 2^n}{n^{100} + 3 \cdot 2^n}$;

Solution: We have

$$\frac{2n^2 + 2^n}{n^{100} + 3 \cdot 2^n} = \frac{2n^2(1/2)^n + 1}{n^{100}(1/2)^n + 3}.$$

Using elementary limits from lectures we know that $n^2(1/2)^n \to 0$ as $n \to \infty$. Similarly, $n^{100}(1/2)^n \to 0$. Therefore, by the limit laws

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2n^2(1/2)^n + 1}{n^{100}(1/2)^n + 3} = \frac{1}{3}.$$

(b) $x_n = \big(n - \sqrt{n(n-1)}\big)$;

Solution: We rewrite x_n as follows:

$$\begin{aligned} x_n &= \big(n - \sqrt{n(n-1)}\big) = \sqrt{n}\big(\sqrt{n} - \sqrt{n-1}\big) \\ &= \sqrt{n}\frac{(\sqrt{n} - \sqrt{n-1})(\sqrt{n} + \sqrt{n-1})}{\sqrt{n} + \sqrt{n-1}} = \frac{\sqrt{n}(n - (n-1))}{\sqrt{n} + \sqrt{n-1}} \\ &= \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n-1}} = \frac{1}{1 + \sqrt{1 - 1/n}}. \end{aligned}$$

Now by the limit laws we have $\displaystyle\lim_{n\to\infty} \frac{1}{1 + \sqrt{1 - 1/n}} = \frac{1}{2}$.

(c) $x_n = \dfrac{1 + 2 + \cdots + n}{n+2} - \dfrac{n}{2}$;

Solution: By induction it is easily proved that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Hence we get

$$x_n = \frac{n(n+1)}{2(n+2)} - \frac{n}{2} = \frac{n}{2}\Big(\frac{n+1}{(n+2)} - 1\Big) = \frac{n}{2}\frac{-1}{(n+2)} = -\frac{1}{2}\frac{1}{(1 + 2/n)}.$$

Using the limit laws we conclude that $x_n \to -1/2$ as $n \to \infty$.

(d) $x_n = (a^n + b^n)^{1/n}$ if $0 \le a \le b$;

Solution: We obviously have

$$b = (b^n)^{1/n} \le (a^n + b^n)^{1/n} \le 2^{1/n} b$$

for all $n \in \mathbb{N}$. We know that $2^{1/n} \to 1$ as $n \to \infty$, so by the squeeze law $x_n \to b$.

3. (a) Let $a > 0$. Define a sequence recursively by choosing $x_0 > 0$ arbitrary and setting $x_{n+1} := \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big)$ for all $n \in \mathbb{N}$. Prove that $(x_n)_{n\ge 1}$ is decreasing and that $x_n \to \sqrt{a}$ as $n \to \infty$. (This is a practical numerical method to compute square roots as (x_n) converges very fast.)

Solution: We prove that $x_{n+1} \le x_n$ for $n \ge 1$. To do so we use the inequality between the geometric and algebraic means of positive numbers α, β

$$\alpha\beta \le \Big(\frac{\alpha+\beta}{2}\Big)^2,$$

which follows since $0 \le (\alpha - \beta)^2$. Hence, setting $\alpha = x_n$ and $\beta := a/x_n$ we get

$$x_{n+1}^2 = \Big(\frac{x_n + a/x_n}{2}\Big)^2 \ge x_n \frac{a}{x_n} = a$$

for all $n \in \mathbb{N}$. Thus $x_n^2 \ge a$ and so

$$x_{n+1}^2 = \Big(\frac{x_n + a/x_n}{2}\Big)^2 \le \Big(\frac{x_n + x_n^2/x_n}{2}\Big)^2 = x_n^2.$$

This proves that $x_{n+1} \le x_n$ for all $n \ge 1$. Note that possibly $x_0 < x_1$ but we only need that $x_1 \ge x_2 \ge x_3, \dots$. Clearly $0 \le x_n$ for all $n \in \mathbb{N}$, so (x_n) is a monotone bounded sequence. Hence it converges to some limit $x \ge 0$. We next determine the limit x. If $x_n \to 0$ then

$$x_{n+1} := \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big) \to \infty$$

which is a contradiction as $x_{n+1} \to 0$ if $x_n \to 0$. Hence $x > 0$. Now by the limit laws

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big) \to \frac{1}{2}\Big(x + \frac{a}{x}\Big) = x.$$

Solving the last equation for x we get $x^2 = a$, that is, $x = \sqrt{a}$.
Note: The above is actually a proof that $\sqrt{a}$ exists for all $a > 0$.

(b) Set $a = x_0 = 2$ and use (a) to compute x_3. Compare the result to $\sqrt{2}$ from your calculator.

Solution: $x_0 = 2$, $x_1 = 3/2$, $x_2 = 17/12$ and $x_3 = 1.4142157$. Now $\sqrt{2} = 1.4142136$ from the calculator. Hence the answer is correct to five digits already!

Extra questions for further practice

4. Let $a \in \mathbb{C}$ with $a \neq 1$. Use induction by n to prove that

$$s_n := \sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}$$

for all $n \in \mathbb{N}$. Conclude that (s_n) converges if and only if $|a| < 1$. In case of convergence show that $s_n \to (1-a)^{-1}$ as $n \to \infty$.

Solution: If $n = 0$ then obviously

$$s_0 = a^0 = 1 = \frac{1 - a^{0+1}}{1 - a}.$$

Hence assume the formula holds for n (induction hypothesis). We need to show that the formula holds for $n+1$. Now, using the definition of s_n and the induction hypothesis

$$s_{n+1} = s_n + a^{n+1} = \frac{1-a^{n+1}}{1-a} + a^{n+1} = \frac{1-a^{n+1}+a^{n+1}(1-a)}{1-a}$$
$$= \frac{1-a^{n+1}+a^{n+1}-a^{n+2}}{1-a} = \frac{1-a^{(n+1)+1}}{1-a},$$

which is exactly what we want. We now look at convergence of (s_n). Note that (s_n) converges if and only if a^n converges and $a \neq 1$. Now we know that $|a^n| = |a|^n$ converges to 0 if and only if $|a| < 1$ (see lectures). If $|a| = 1$ and $a \neq 1$ then a^n diverges as $a^{n+1} - a^n \not\to 0$ (the argument of a is non-zero, so we have a rotation by a fixed angle!). If $a = 1$ then clearly (s_n) diverges as in that case (s_n) is unbounded. Hence (s_n) converges if and only if $|a| < 1$. By the limit laws

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} a^k = \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a}.$$

Challenge questions (optional)

***5.** Consider the sequence given by $s_n := \sum_{k=0}^{n} \frac{1}{k!}$. In lectures it is shown that $s_n \to e$.

(a) Prove that $s_n < e < s_n + \dfrac{1}{n!n}$ for all $n \geq 1$. Determine e to four decimal places.

Solution: We have to estimate the error term

$$0 < r_n := e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}$$

For $k \geq n+1$ we have

$$k! = \underbrace{k \cdot (k-1) \cdot \ldots (n+1)}_{k-n \text{ terms}} \cdot \underbrace{n \cdot \cdots 2 \cdot 1}_{n \text{ terms}} \geq (n+1)^{k-n} n!,$$

with strict inequality if $k > n+1$. Hence by using the formula for the geometric series

$$r_n < \frac{1}{n!} \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n}} = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n!(n+1)} \cdot \frac{1}{1-\frac{1}{n+1}} = \frac{1}{n!n}.$$

Hence $s_n < e < s_n + \dfrac{1}{n!n}$ for all $n \geq 1$ as claimed. We next approximate e. We make sure the error $r_n < 10^{-4}$. This is the case when $1/(n!n) < 10^{-4}$. If we choose $n = 7$, then $r_7 \leq 1/(7!7) \approx 2.8345 \cdot 10^{-5}$. Evaluating s_7 on a calculator we get

$$2.718253 \leq e \leq 2.718283.$$

(b) Use (a) to show that e is irrational. (Give a proof by contradiction.)

Solution: Assume that $e = \dfrac{m}{n}$ for some $m, n \in \mathbb{N}$. From (a) we know that $2 < e < 3$, so $m \geq n \geq 2$. Again by part (a)

$$z := n!\, s_n < \frac{n!\, m}{n} = (n-1)!\, m < n!\, s_n + \frac{1}{n} = z + \frac{1}{n}.$$

Now observe that

$$z = n!\, s_n = \sum_{k=0}^{n} \frac{n!}{k!} = 1 + \sum_{k=0}^{n-1} n(n-1)\ldots(k+1) \in \mathbb{N}$$

Clearly $(n-1)!\, m \in \mathbb{N}$, but this is not possible since $z < (n-1)!\, m < z + 1/n \leq z + 1/2$. Hence e is irrational.