THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 3 (Week 4)

MATH2962: Real and Complex Analysis (Advanced) — Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Consider the sequence $x_n = \dfrac{3(-1)^n n^2}{n^2 - n + 1}$ for $n \geq 0$.

(a) Find $a_n = \inf_{k \geq n} x_k$ and $b_n = \sup_{k \geq n} x_k$.

(b) Hence compute $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$.

2. Compute the limit inferior and limit superior of the following sequences.

(a) $x_n = \begin{cases} n & \text{if } n \text{ is even} \\ 1/n & \text{if } n \text{ is odd} \end{cases}$

(b) $x_n = \begin{cases} (-1)^{n/2} \frac{n}{n+1} & \text{if } n \text{ is even} \\ \frac{n^2-1}{2n^2+1} & \text{if } n \text{ is odd} \end{cases}$

(c) $\dfrac{3n + (-1)^n}{5n - 1}$

(d) $s_n = \displaystyle\sum_{k=0}^{n} (-1)^k$

3. Let (x_n) and (y_n) be *bounded* sequences in $\mathbb{R}$.

(a) Prove that
$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

(b) Prove that
$$\limsup_{n\to\infty}(x_n + y_n) = \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$
if at least one of the sequences converges.

(c) By giving a counter example, show that strict inequality in (a) is possible.

Extra questions for further practice

4. Let (x_n) and (y_n) be *bounded* sequences in $\mathbb{R}$ with $x_n, y_n \geq 0$ for all $n \in \mathbb{N}$. We know from lectures that
$$\limsup_{n\to\infty}(x_n y_n) \leq \Big(\limsup_{n\to\infty} x_n\Big)\Big(\limsup_{n\to\infty} y_n\Big).$$
By giving a counter example, show that strict inequality is possible.

Challenge questions (optional)

5. *(a) Suppose that (a_n) is a sequence in $\mathbb{R}$ with $a_n \neq 0$ for all $n \in \mathbb{N}$. If $(a_{n+1}/a_n)_{n\in\mathbb{N}}$ is a bounded sequence, prove that
$$\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \leq \liminf_{n\to\infty} \sqrt[n]{|a_n|} \leq \limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

(b) Use part (a) to compute the limit of $x_n = \sqrt[n]{n!}/n$ as $n \to \infty$.