

Solutions to Tutorial 3 (Week 4)

MATH2962: Real and Complex Analysis (Advanced)

Semester 1, 2012

Web Page: <http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/>

Lecturer: Florica Cirstea

Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Consider the sequence $x_n = \frac{3(-1)^n n^2}{n^2 - n + 1}$ for $n \geq 0$.

(a) Find $a_n = \inf_{k \geq n} x_k$ and $b_n = \sup_{k \geq n} x_k$.

Solution: Using that $x_n < 0$ for $n \geq 1$ odd and $x_n > 0$ for $n \geq 2$ even, we obtain that

$$\begin{cases} a_n = \inf_{\substack{k \geq n \\ k \text{ odd}}} x_k = - \sup_{\substack{k \geq n \\ k \text{ odd}}} (-x_k) = - \sup_{\substack{k \geq n \\ k \text{ odd}}} |x_k|, \\ b_n = \sup_{\substack{k \geq n \\ k \text{ even}}} x_k = \sup_{\substack{k \geq n \\ k \text{ even}}} |x_k|. \end{cases}$$

For this reason, we want to study the monotonicity of $|x_n|$. Note that

$$x_n = \frac{3(-1)^n n^2}{n^2 - n + 1} = \frac{3(-1)^n}{1 - \frac{1}{n} + \frac{1}{n^2}} = \frac{3(-1)^n}{(\frac{1}{n} - \frac{1}{2})^2 + \frac{3}{4}}$$

for all $n \geq 1$. Since $(\frac{1}{n} - \frac{1}{2})^2$ is increasing for $n \geq 2$, it follows that $|x_n|$ is decreasing for $n \geq 2$. Moreover, $x_0 = 0$, $x_1 = -3$, $x_2 = 4$ and $x_3 = -27/7$. Hence

$$a_n = \begin{cases} -27/7 & \text{for } n = 0, 1, \\ x_{n+1} & \text{for } n \geq 2 \text{ even,} \\ x_n & \text{for } n \geq 3 \text{ odd,} \end{cases} \quad b_n = \begin{cases} 4 & \text{for } n = 0, 1, \\ x_n & \text{for } n \geq 2 \text{ even,} \\ x_{n+1} & \text{for } n \geq 3 \text{ odd.} \end{cases}$$

(b) Hence compute $\liminf_{n \rightarrow \infty} x_n$ and $\limsup_{n \rightarrow \infty} x_n$.

Solution: Notice that

$$|x_n| = \frac{3}{1 - \frac{1}{n} + \frac{1}{n^2}} \rightarrow 3 \text{ as } n \rightarrow \infty.$$

Thus we get $\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{2n+1} = -3$ and $\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{2n} = 3$.

2. Compute the limit inferior and limit superior of the following sequences.

(a) $x_n = \begin{cases} n & \text{if } n \text{ is even} \\ 1/n & \text{if } n \text{ is odd} \end{cases}$

Solution: We see that $\lim_{n \rightarrow \infty} x_{2n} = \infty$ and $\lim_{n \rightarrow \infty} x_{2n+1} = 0$. Hence, any subsequence of (x_n) which has a limit will either tend to ∞ or to 0 as $n \rightarrow \infty$. It follows that $\liminf_{n \rightarrow \infty} x_n = 0$ and $\limsup_{n \rightarrow \infty} x_n = \infty$. Recall that when (x_n) is not bounded from above, then $\limsup_{n \rightarrow \infty} x_n = \infty$.

(b) $x_n = \begin{cases} (-1)^{n/2} \frac{n}{n+1} & \text{if } n \text{ is even} \\ \frac{n^2-1}{2n^2+1} & \text{if } n \text{ is odd} \end{cases}$

Solution: We clearly have

$$0 \leq \frac{n^2 - 1}{2n^2 + 1} \leq \frac{n^2}{2n^2} = \frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leq \frac{n}{n+1} \leq 1$$

for all $n \geq 1$. Moreover $\frac{n}{n+1}$ is increasing in n . Hence

$$a_n = \inf_{k \geq n} x_k = -1 \quad \text{and} \quad b_n = \sup_{k \geq n} x_k = 1$$

for all $n \in \mathbb{N}$. Therefore $\liminf_{n \rightarrow \infty} x_n = -1$ and $\limsup_{n \rightarrow \infty} x_n = 1$.

We now give another method of finding $\liminf_{n \rightarrow \infty} x_n$ and $\limsup_{n \rightarrow \infty} x_n$. Notice that $x_{4n} \rightarrow 1$ and $x_{4n+2} \rightarrow -1$ as $n \rightarrow \infty$. On the other hand, $x_{2n+1} \rightarrow 1/2$ as $n \rightarrow \infty$ (which means that (x_{4n+1}) and (x_{4n+3}) converge to $1/2$). Hence, if a subsequence of (x_n) has a limit it must be either 1, or $1/2$ or -1 . Since $\liminf_{n \rightarrow \infty} x_n$ corresponds to the smallest among these values and $\limsup_{n \rightarrow \infty} x_n$ to the biggest, we find that

$$\liminf_{n \rightarrow \infty} x_n = -1, \quad \limsup_{n \rightarrow \infty} x_n = 1.$$

(c) $\frac{3n + (-1)^n}{5n - 1}$

Solution: We have

$$\frac{3n + (-1)^n}{5n - 1} = \frac{3 + (-1)^n/n}{5 - 1/n} \rightarrow \frac{3}{5}.$$

Since the limit exists it also equals the limit inferior and the limit superior.

(d) $s_n = \sum_{k=0}^n (-1)^k$

Solution: Notice that $s_0 = 1$ and $s_{n+1} = s_n + (-1)^{n+1}$ for $n \geq 0$. Hence, we have

$$\begin{aligned} s_0 &= 1 \\ s_1 &= s_0 - 1 = 0 \\ s_2 &= s_1 + 1 = 1. \\ s_3 &= s_2 - 1 = 0. \end{aligned}$$

By induction after n , it can be proved that $s_{2n} = 1$ and $s_{2n+1} = 0$ for every $n \geq 0$. Hence $\liminf_{n \rightarrow \infty} s_n = 0$ and $\limsup_{n \rightarrow \infty} s_n = 1$.

3. Let (x_n) and (y_n) be *bounded* sequences in \mathbb{R} .

(a) Prove that

$$\limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n.$$

Solution: Clearly $x_\ell + y_\ell \leq \sup_{k \geq n} x_k + \sup_{k \geq n} y_k$ for all $\ell \geq n$. Hence by definition of a supremum

$$\sup_{\ell \geq n} (x_\ell + y_\ell) \leq \sup_{k \geq n} x_k + \sup_{k \geq n} y_k$$

for all $n \in \mathbb{N}$. By definition of the limit superior and the limit laws

$$\limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n.$$

(b) Prove that

$$\limsup_{n \rightarrow \infty} (x_n + y_n) = \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n$$

if at least one of the sequences converges.

Solution: Suppose that (x_n) converges (just rename the sequences if (y_n) converges). We know from lectures that

$$\limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n. \quad (1)$$

Applying the same fact we know that

$$\limsup_{n \rightarrow \infty} y_n = \limsup_{n \rightarrow \infty} (x_n + y_n - x_n) \leq \limsup_{n \rightarrow \infty} (x_n + y_n) + \limsup_{n \rightarrow \infty} (-x_n). \quad (2)$$

Since we have

$$\limsup_{n \rightarrow \infty} (-x_n) = \lim_{n \rightarrow \infty} (-x_n) = - \lim_{n \rightarrow \infty} x_n, \quad (3)$$

we conclude from (2) and (3) that

$$\limsup_{n \rightarrow \infty} y_n + \lim_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} (x_n + y_n). \quad (4)$$

Combining (1) and (4) we get the required identity.

Remark. Note that (3) is not valid if (x_n) does not have a limit because then

$$\limsup_{n \rightarrow \infty} (-x_n) = - \liminf_{n \rightarrow \infty} x_n > - \limsup_{n \rightarrow \infty} x_n.$$

(c) By giving a counter example, show that strict inequality in (a) is possible.

Solution: From (b) above, we see that the strict inequality in (a) could only occur when both (x_n) and (y_n) don't converge (that is, (x_n) and (y_n) don't have a limit since they are assumed bounded). Choose, for example,

$$x_n := (-1)^n \quad \text{and} \quad y_n = (-1)^{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Then $\limsup_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} y_n = 1$. On the other hand, $x_n + y_n = (-1)^n(1 - 1) = 0$ for all $n \in \mathbb{N}$, so $\limsup_{n \rightarrow \infty} (x_n + y_n) = 0$. Hence,

$$0 = \limsup_{n \rightarrow \infty} (x_n + y_n) < \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n = 1 + 1 = 2.$$

Remark. More general examples can be given by looking for sequences of the form

$$x_n = \begin{cases} a_1 & \text{if } n \text{ is even,} \\ a_2 & \text{if } n \text{ is odd,} \end{cases} \quad y_n = \begin{cases} b_1 & \text{if } n \text{ is even,} \\ b_2 & \text{if } n \text{ is odd.} \end{cases}$$

We have $\limsup_{n \rightarrow \infty} x_n = \max\{a_1, a_2\}$ and $\limsup_{n \rightarrow \infty} y_n = \max\{b_1, b_2\}$. Moreover, $\limsup_{n \rightarrow \infty} (x_n + y_n) = \max\{a_1 + b_1, a_2 + b_2\}$. By choosing a_1, a_2, b_1 and b_2 in \mathbb{R} with

$$\max\{a_1 + b_1, a_2 + b_2\} < \max\{a_1, a_2\} + \max\{b_1, b_2\},$$

we obtain a strict inequality in (a).

Extra questions for further practice

4. Let (x_n) and (y_n) be *bounded* sequences in \mathbb{R} with $x_n, y_n \geq 0$ for all $n \in \mathbb{N}$. We know from lectures that

$$\limsup_{n \rightarrow \infty} (x_n y_n) \leq \left(\limsup_{n \rightarrow \infty} x_n \right) \left(\limsup_{n \rightarrow \infty} y_n \right).$$

By giving a counter example, show that strict inequality is possible.

Solution: We set $x_n := 2 + (-1)^n$ and $y_n = 2 + (-1)^{n+1}$ for all $n \in \mathbb{N}$. We then have $x_n, y_n > 0$ and $\limsup_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} y_n = 3$. On the other hand, $x_n y_n = (2 + (-1)^n)(2 - (-1)^n) = 3$ for all $n \in \mathbb{N}$, so $\limsup_{n \rightarrow \infty} (x_n y_n) = 3$. Hence,

$$3 = \limsup_{n \rightarrow \infty} (x_n y_n) < \left(\limsup_{n \rightarrow \infty} x_n \right) \left(\limsup_{n \rightarrow \infty} y_n \right) = 3 \cdot 3 = 9.$$

Remark. If either (x_n) or (y_n) has a limit as $n \rightarrow \infty$, then equality holds. Suppose that (x_n) has a limit, then it will *converge* to a non-negative number x (since (x_n) is bounded and $x_n \geq 0$). If $x > 0$, then from lectures, we know that equality holds. If $x = 0$, then $(x_n y_n)$ converges to 0 since (y_n) is bounded. Hence, again we have equality. So, to have a strict inequality, we must look at cases when both (x_n) and (y_n) don't have a limit as $n \rightarrow \infty$. Such situations can be found using the sequences in Remark for Question 3(c) and choosing a_1, a_2, b_1 and b_2 in \mathbb{R} such that $\max\{a_1 b_1, a_2 b_2\} < \max\{a_1, a_2\} \max\{b_1, b_2\}$.

Challenge questions (optional)

5. *(a) Suppose that (a_n) is a sequence in \mathbb{R} with $a_n \neq 0$ for all $n \in \mathbb{N}$. If $(a_{n+1}/a_n)_{n \in \mathbb{N}}$ is a bounded sequence, prove that

$$\liminf_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq \limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}.$$

Solution: We only prove the last inequality. The middle one is obvious from the definition of the limit superior and inferior, and the first one can be obtained by reversing the signs. Since $(a_{n+1}/a_n)_{n \in \mathbb{N}}$ is bounded, we have that

$$s := \limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$$

exists in \mathbb{R} . Then the definition of the limit superior gives that for every $\varepsilon > 0$, there exists $N \geq 1$ (where N depends on ε) such that

$$\sup_{k \geq n} \frac{|a_{k+1}|}{|a_k|} < s + \varepsilon \quad \text{for all } n \geq N.$$

Hence, $|a_{k+1}|/|a_k| < s + \varepsilon$ for all $k \geq N$. Hence if $n > N$, then by multiplying all the inequalities corresponding to $k = n - 1, \dots, N$, we obtain that

$$\frac{|a_n|}{|a_N|} = \frac{|a_n|}{|a_{n-1}|} \cdot \frac{|a_{n-1}|}{|a_{n-2}|} \cdot \dots \cdot \frac{|a_{N+1}|}{|a_N|} \leq (s + \varepsilon)^{n-N}.$$

Therefore $\sqrt[n]{|a_n|} \leq (s + \varepsilon)^{1-N/n} \sqrt[n]{|a_N|}$ for all $n > N$. We know that $\sqrt[n]{|a_N|} \rightarrow 1$ as $n \rightarrow \infty$ and similarly $(s + \varepsilon)^{1-N/n} \rightarrow 1$ as $n \rightarrow \infty$. Hence, by the limit laws for the limit superior

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq s + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we get the required inequality by letting $\varepsilon \rightarrow 0$.

- (b) Use part (a) to compute the limit of $x_n = \sqrt[n]{n!}/n$ as $n \rightarrow \infty$.

Solution: We have $x_n = \sqrt[n]{a_n}$ if we define

$$a_n := \frac{n!}{n^n}.$$

Note that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \left(\frac{n}{n+1} \right)^n = \left(1 + \frac{1}{n} \right)^{-n} \rightarrow \frac{1}{e}.$$

From part (a) and the squeeze law, we conclude that

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{e}.$$