THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 4 (Week 5)

MATH2962: Real and Complex Analysis (Advanced) — Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. (a) Suppose that $f\colon [1,\infty) \to \mathbb{R}$ is a positive decreasing function. Show that for every $n \geq 2$

$$f(2) + \cdots + f(n) \leq \int_1^n f(x)\,dx \leq f(1) + f(2) + \cdots + f(n-1). \tag{1}$$

(b) Assume that $f(x)$ is a positive decreasing function on $[1,\infty)$. Using (a), establish the following *Integral Test*: The series $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if

$$\int_1^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_1^n f(x)\,dx < \infty. \tag{2}$$

(c) Use the Integral Test to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

2. Determine which of the series below converge, and which diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{2n^2+n+1}$; (c) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$; (e) $\sum_{n=0}^{\infty} \frac{2^n-1}{3^n+1}$; (g) $\sum_{n=2}^{\infty} \frac{1}{n^2\log n}$;

(b) $\sum_{n=1}^{\infty} \frac{1}{2n^2-n+1}$; (d) $\sum_{n=1}^{\infty} \frac{1}{1+3\sqrt{n}}$; (f) $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n-1}$; (h) $\sum_{n=1}^{\infty} \frac{\log n}{n}$;

3. Consider two sequences (a_n) and (b_n) in $\mathbb{R}$ or $\mathbb{C}$ with $b_n \neq 0$ for all $n \in \mathbb{N}$. We call the sequence (a_n) equivalent to (b_n) if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

If that is the case, we write $a_n \sim b_n$.

(a) Show that equivalence of sequences is an equivalence relation, that is, it has the following properties:

(i) $a_n \sim a_n$ (reflexivity);

(ii) $a_n \sim b_n \Rightarrow b_n \sim a_n$ (symmetry);

(iii) $a_n \sim b_n$ and $b_n \sim c_n$ implies that $a_n \sim c_n$ (transitivity).

(b) Suppose that $a_n \sim b_n$. Show that (a_n) converges if and only if (b_n) converges. In case of convergence show that the limits of (a_n) and (b_n) are the same.

Extra questions for further practice

4. Let $a_k, b_k > 0$ for all $k \in \mathbb{N}$ and suppose that $a_k \sim b_k$. Show that $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} b_k$ converges.

5. Prove that the following sequences are equivalent and decide whether they converge or diverge.

(a) $\dfrac{4n+1}{3n-1} \sim \dfrac{4}{3}$

(b) $\dfrac{3^n + 2^n}{3^n - 2^n} \sim 1$

(c) $\dfrac{\sqrt{n^2 + 3^n}}{n^4 + 3n + 1} \sim \dfrac{3^{n/2}}{n^4}$

*(d) $\ln n \sim s_n := \displaystyle\sum_{k=1}^{n} \frac{1}{k}$

Challenge questions (optional)

***6.** Suppose that $f \colon \mathbb{K}^N \to \mathbb{K}^N$ is a function such that there exists $L \in (0,1)$ with

$$\|f(x) - f(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in \mathbb{K}^N. \tag{3}$$

(a) Given $x_0 \in \mathbb{K}^N$ and $x_{n+1} := f(x_n)$ for any $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.

(b) Show that there is a unique point in $\mathbb{K}^N$ such that $x = f(x)$.