Material marked with * are more difficult questions.

**Material covered**

(1) Series and their convergence, comparison tests
(2) Cauchy sequences and convergence

**Outcomes**

This tutorial helps you to

(1) have solid foundations in the more formal aspects of analysis, including a knowledge of precise definitions, how to apply them and the ability to write simple proofs;
(2) be able to work with inequalities, limits and limit inferior/superior;
(3) know and be able to apply convergence tests for sequences and series;

**Summary of essential material**

Series are sequences formed by summing terms $a_k \in \mathbb{K}^N$. We denote series by

$$\sum_{k=0}^{\infty} a_k.$$ 

We call

- $s_n := \sum_{k=0}^{n}$ then $n$-th partial sum of the series.
- $(s_n)$ the sequence of partial sums

We say that the series $\sum_{k=0}^{\infty} a_k$

- is **convergent** if the sequence of partial sums $(s_n)$ converges;
- is **divergent** if the sequence of partial sums $(s_n)$ diverges.

Consider the class of series with **non-negative terms**, that is, $a_k \geq 0$ for all $n \in \mathbb{N}$. Then the sequence of partial sums $(s_n)$ is increasing, and hence

$$(s_n)$$ converges if and only if it is bounded.

That fact is the basis for the comparison tests:
Comparison test. Suppose that there exists \( m \in \mathbb{N} \) so that \( 0 \leq a_k \leq b_k \) for all \( k \geq m \).

- If \( \sum_{k=0}^{\infty} b_k \) converges, then \( \sum_{k=0}^{\infty} a_k \) converges.
- If \( \sum_{k=0}^{\infty} a_k \) diverges, then \( \sum_{k=0}^{\infty} b_k \) diverges.

Limit comparison test. Suppose that there exists \( m \in \mathbb{N} \) so that \( 0 \leq a_k \) and \( 0 < b_k \) for all \( k \geq m \), and that

\[ \limsup_{n \to \infty} \frac{a_n}{b_n} < \infty \quad \text{or equivalently} \quad \left( \frac{a_n}{b_n} \right)_{n \in \mathbb{N}} \text{ is bounded} \]

- If \( \sum_{k=0}^{\infty} b_k \) converges, then \( \sum_{k=0}^{\infty} a_k \) converges.
- If \( \sum_{k=0}^{\infty} a_k \) diverges, then \( \sum_{k=0}^{\infty} b_k \) diverges.

Questions to complete during the tutorial

1. (a) Suppose that \( f : [1, \infty) \to \mathbb{R} \) is a positive decreasing function and set \( a_n := f(n) \) for \( n \in \mathbb{N} \). For \( n \geq 1 \) define

\[ s_n := \sum_{k=1}^{n} a_k \quad \text{and} \quad I_n := \int_{1}^{n} f(x) \, dx. \]

(i) Use upper and lower Riemann sums for the integral \( I_n \) to show that, for all \( n \geq 2 \),

\[ s_n - a_1 \leq I_n \leq s_n - a_n. \]

**Solution:** Let \( n \geq 2 \). Then the integrals \( I_n \) exist as \( f \) is monotone. We partition the interval \([1, n]\) into \( n - 1\) sub-intervals of length 1. Because \( f \) is a decreasing function, the upper and lower Riemann sums for this partition are \( a_1 + a_2 + \cdots + a_{n-1} \) and \( a_2 + \cdots + a_n \), respectively. We illustrate that on the graph below.
Hence we get the inequality
\[ s_n - a_1 = a_2 + \cdots + a_n \leq \int_1^n f(x) \, dx \leq a_1 + a_2 + \cdots + a_{n-1} = s_n - a_n \]
for all \( n \geq 2 \).

(ii) Hence establish the Integral Test: Suppose that \( a_n = f(n) \) where \( f(x) \) is positive decreasing on \([1, \infty)\). Then \( \sum_{n=1}^\infty a_n \) is convergent if and only if
\[ \int_1^\infty f(x) \, dx := \lim_{n \to \infty} \int_1^n f(x) \, dx < \infty. \]

**Solution:** Because \( f \) is assumed positive, \( I_n \) is an increasing sequence, and therefore convergent if and only if it is bounded. Similarly, the sequence of partial sums \((s_n)\) is increasing, and therefore convergent if and only if it is bounded.
Assume first that \((I_n)\) is bounded and therefore convergent. Hence assume that
\[ I := \lim_{n \to \infty} I_n = \int_1^\infty f(x) \, dx < \infty \]
From the first inequality in the previous part
\[ s_n \leq a_1 + I_n \leq a_1 + I < \infty \]
for all \( n \geq 2 \). Hence the sequence of partial sums \( s_n \) is bounded, so \( s_n \) converges.
Assume now that \( I_n \) is unbounded, then by the second inequality in the previous part
\[ I_n \leq s_n \]
for all \( n \geq 2 \). Therefore \((s_n)\) is unbounded as well, so diverges.

(b) Use the integral test to show that \( \sum_{n=1}^\infty \frac{1}{n^p} \) is convergent if and only if \( p > 1 \).

**Solution:** If \( p \leq 0 \), then \( 1/n^p \not\to 0 \) as \( n \to \infty \), and therefore the series diverges.
If \( p > 0 \), then \( 1/x^p \) is decreasing to zero, and therefore convergence is possible. We distinguish some cases. If \( p = 1 \) then
\[ I_n = \int_1^n \frac{1}{x^p} \, dx = \log x \bigg|_1^n = \log n \xrightarrow{n \to \infty} \infty. \]
Hence \( I_n \) diverges. If \( p \neq 1 \), then
\[ I_n = \int_1^n \frac{1}{x^p} \, dx = \frac{x^{1-p}}{1-p} \bigg|_1^n = \frac{n^{1-p} - 1}{1-p}. \]
Hence \((I_n)\) converges if \( p > 1 \), and diverges if \( p \in (0, 1) \). From the Integral Test we conclude that \((s_n)\) converges if and only if \( p > 1 \).

2. Determine which of the series below converge, and which diverge.
(a) \[ \sum_{n=1}^{\infty} \frac{1}{2n^2 + n + 1}; \]

**Solution:** Clearly \( 0 \leq \frac{1}{2n^2 + n + 1} \leq \frac{1}{2n^2} \). We know that \( \sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, so the original series converges by the comparison test.

(b) \[ \sum_{n=1}^{\infty} \frac{1}{2n^2 - n + 1}; \]

**Solution:** Note that for \( \frac{n^2}{2n^2 - n + 1} \to \frac{1}{2} \) as \( n \to \infty \). Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, so the original series converges by the limit comparison test.

(c) \[ \sum_{n=1}^{\infty} \frac{1}{2n - 1}; \]

**Solution:** Clearly \( \frac{1}{2n - 1} \geq \frac{1}{2n} \) for all \( n \in \mathbb{N} \). As the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, so does the original series by the comparison test.

(d) \[ \sum_{n=1}^{\infty} \frac{1}{1 + 3\sqrt{n}}; \]

**Solution:** Clearly \( \frac{1}{1 + 3\sqrt{n}} \geq \frac{1}{4\sqrt{n}} \geq \frac{1}{4n} \) for all \( n \in \mathbb{N} \). As the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, so does the original series by the comparison test.

(e) \[ \sum_{n=0}^{\infty} \frac{2^n - 1}{3^n + 1}; \]

**Solution:** Clearly \( \frac{2^n - 1}{3^n + 1} \leq \left(\frac{2}{3}\right)^n \) for all \( n \in \mathbb{N} \). Hence the series converges by comparison with the geometric series \( \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \).

(f) \[ \sum_{n=1}^{\infty} \frac{2^n + 1}{3^n - 1}; \]

**Solution:** Note that \( \frac{2^n + 1}{3^n - 1} \to 1 \). Since \( \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \) converges, the original series converges by the limit comparison test.

(g) \[ \sum_{n=2}^{\infty} \frac{1}{n^2 \log n}; \]

**Solution:** Clearly \( \frac{1}{n^2 \log n} \leq \frac{1}{n^2} \) for \( n \geq 3 \), so the series converges by comparison to the convergent series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

(h) \[ \sum_{n=1}^{\infty} \frac{\log n}{n}; \]
Solution: Clearly \( 0 \leq \frac{1}{n} \leq \frac{\log n}{n} \) for \( n \geq 3 \). We know that the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, so the original series diverges by the comparison test.

\[ *(i) \sum_{n=1}^{\infty} \frac{n}{4^n}; \]

Solution: We know that exponential decay is stronger than polynomial growth. Hence we do a limit comparison with the convergent geometric series \( \sum_{n=1}^{\infty} \frac{1}{2^n} \).

We have
\[
\frac{n}{4^n} = \frac{1}{2^n} \to 0
\]
as \( n \to \infty \) by one of the elementary limits. Hence the series converges by the limit comparison test. We will discuss more efficient convergence tests for a sequence such as this one (root and ratio tests).

3. Consider two sequences \((a_n)\) and \((b_n)\) in \( \mathbb{R} \) or \( \mathbb{C} \) with \( b_n \neq 0 \) for all \( n \in \mathbb{N} \). We call the sequence \((a_n)\) equivalent to \((b_n)\) if
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 1.
\]
If that is the case, we write \( a_n \sim b_n \).

(a) Show that equivalence of sequences is an equivalence relation, that is, it has the following properties:

(i) \( a_n \sim a_n \) (reflexivity);

Solution: Clearly \( \frac{a_n}{a_n} = 1 \to 1 \), so \( a_n \sim a_n \).

(ii) \( a_n \sim b_n \Rightarrow b_n \sim a_n \) (symmetry);

Solution: By the limit laws \( \frac{b_n}{a_n} = \frac{1}{2^n} \to 1 \), so \( b_n \sim a_n \) if \( a_n \sim b_n \).

(iii) \( a_n \sim b_n \) and \( b_n \sim c_n \) implies that \( a_n \sim c_n \) (transitivity).

Solution: By the limit laws \( \frac{a_n}{c_n} = \frac{a_n b_n}{b_n c_n} \to 1 \cdot 1 = 1 \).

(b) Suppose that \( a_n \sim b_n \). Show that \((a_n)\) converges if and only if \((b_n)\) converges. In case of convergence show that the limits of \((a_n)\) and \((b_n)\) are the same.

Solution: If \( b_n \to b \), then by the limit laws \( a_n = \frac{a_n b_n}{b_n b_n} \to 1 \cdot b = b \). Hence, \( a_n \to b \). If \( a_n \sim b_n \), then by (a)(ii) we also have \( b_n \sim a_n \). Hence, interchanging the roles of \( a_n \) and \( b_n \) in the above argument we see that \( a_n \to a \) implies that \( b_n \to a \).

Extra questions for further practice

4. Let \( a_k, b_k > 0 \) for all \( k \in \mathbb{N} \) and suppose that \( a_k \sim b_k \). Show that \( \sum_{k=0}^{\infty} a_k \) converges if and only if \( \sum_{k=0}^{\infty} b_k \) converges.
**Solution:** Since $\frac{a_n}{b_n} \to 1$, the convergence of $\sum_{k=0}^\infty b_k$ implies the convergence of $\sum_{k=0}^\infty a_k$ by the limit comparison test. Since also $\frac{b_n}{a_n} \to 1$, the convergence of $\sum_{k=0}^\infty a_k$ implies the convergence of $\sum_{k=0}^\infty b_k$, again by the limit comparison test, completing the proof of the statement.

5. Prove that the following sequences are equivalent and decide whether they converge or diverge.

(a) \[ \frac{4n + 1}{3n - 1} \sim \frac{4}{3} \]

**Solution:** We have \[ \frac{4n + 1}{3n - 1} = \frac{4 + 1/n}{3 - 1/n} \to 1 \]. Since the constant sequence $3/4$ converges, the given sequence converges to $3/4$.

(b) \[ \frac{3^n + 2^n}{3^n - 2^n} \sim 1 \]

**Solution:** We have \[ \frac{3^n + 2^n}{3^n - 2^n} = \frac{1 + (2/3)^n}{1 - (2/3)^n} \to 1 \]. Since the constant sequence $1$ converges, the given sequence converges to $1$.

(c) \[ \frac{\sqrt{n^2 + 3^n}}{n^4 + 3n + 1} \sim \frac{3^{n/2}}{n^4} \]

**Solution:** We have \[ \frac{\sqrt{n^2 + 3^n}}{n^4 + 3n + 1} = \frac{\sqrt{n^2 + 3^n}}{n^4 + 3n + 1} \to 1 \]. Since the sequence $3^{n/2}/n^4$ diverges, the given sequence diverges.

*(d) \[ \ln n \sim s_n := \sum_{k=1}^{n} \frac{1}{k} \]

**Solution:** For $n \geq 2$ we can write \[ \ln n = \int_1^n \frac{1}{x} \, dx \]

Using the estimate in Question 1(a) we get \[ s_n - 1 = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \ln n \]

\[ = \int_1^n \frac{1}{x} \, dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n - 1} = s_n - \frac{1}{n} \]

for all $n \geq 2$. Hence, \[ 1 - \frac{1}{s_n} \leq \frac{\ln n}{s_n} \leq 1 - \frac{1}{ns_n} \]

for all $n \geq 2$. Using that $s_n \to \infty$ (divergence of harmonic series) and the squeeze law, we deduce that \[ \lim_{n \to \infty} \frac{\ln n}{s_n} = 1. \]

Therefore $\ln n \sim s_n$ as claimed. This shows that the partial sums of the harmonic series grow like $\ln n$. 


6. Let \((x_n)\) be a sequence in \(\mathbb{R}^N\) and \((a_n)\) a sequence in \(\mathbb{R}\) with \(a_n \geq 0\) for all \(n \in \mathbb{N}\). Assume that \(\|x_{n+1} - x_n\| \leq a_n\) for all \(n \in \mathbb{N}\), and that the series \(\sum_{k=0}^{\infty} a_k\) converges. Show that \((x_n)\) is convergent.

**Hint:** Rewrite the sequence as a telescoping sum and apply the Cauchy criterion.

**Solution:** We first note that
\[
x_{n+1} - x_0 = (x_{n+1} - x_n) + (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - x_0)
\]
\[
= \sum_{k=0}^{n} (x_{k+1} - x_k)
\]
and hence
\[
x_{n+1} = x_0 + \sum_{k=0}^{n} (x_{k+1} - x_k).
\]

We have therefore written the sequence \((x_n)\) as a series, and can therefore attempt to use the Cauchy criterion for series. We note that from the assumptions
\[
\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{\infty} a_k < \infty
\]
for all \(n \geq 0\). As \(\sum_{k=0}^{\infty} a_k\) is convergent the Cauchy criterion implies that for every \(\varepsilon > 0\) there exists \(n_0 \in \mathbb{N}\) such that
\[
0 \leq \sum_{k=n+1}^{m} a_k < \varepsilon
\]
for all \(m > n > n_0\). Hence, using the triangle inequality and the assumptions,
\[
\left\| \sum_{k=n+1}^{m} (x_{k+1} - x_k) \right\| \leq \sum_{k=n+1}^{m} \|x_{k+1} - x_k\| \leq \sum_{k=n+1}^{m} a_k < \varepsilon
\]
for all \(m > n > n_0\). Therefore the series \(\sum_{k=0}^{\infty} (x_{k+1} - x_k)\) satisfies the Cauchy criterion and hence converges. We therefore conclude that
\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} x_{n+1} = x_0 + \sum_{k=0}^{\infty} (x_{k+1} - x_k)
\]
exists.

**Challenge questions (optional)**

*7.* Suppose that \(f: \mathbb{K}^N \to \mathbb{K}^N\) is a function such that there exists \(L \in (0, 1)\) with
\[
\|f(x) - f(y)\| \leq L \|x - y\|
\]
for all \(x, y \in \mathbb{K}^N\). A function with that property is called a contraction since any pair of image points is closer together than the original points.

(a) Let \(x_0 \in \mathbb{K}^N\) and define \(x_{n+1} := f(x_n)\) for \(n \geq 0\). Show that \((x_n)\) is a Cauchy sequence.

**Solution:** First consider the increments \(\|x_{n+1} - x_n\|\). From the definition of the sequence and the property of \(f\)
\[
\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1})\| \leq L \|x_n - x_{n-1}\| \quad (1)
\]
for all \( n \geq 1 \). If we iterate this estimate starting from \( n = 1 \) we expect that
\[
\|x_{n+1} - x_n\| \leq L^n \|x_1 - x_0\| \tag{2}
\]
for all \( n \in \mathbb{N} \). We give a proper proof by induction. For \( n = 1 \) the statement is obvious from (1) with \( n = 1 \). Suppose now the statement is true for some \( n \geq 1 \). Then by (1) and the induction hypothesis
\[
\|x_{n+2} - x_{n+1}\| \leq L \|x_{n+1} - x_n\| \leq L L^n \|x_1 - x_0\| = L^{n+1} \|x_1 - x_0\|,
\]
so the statement is true for \( n + 1 \). We use (2) to show that \((x_n)\) is a Cauchy sequence. In particular, using the formula for the partial sum of a geometric series
\[
\|x_m - x_n\| \leq \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| \leq \|x_1 - x_0\| \sum_{k=n}^{m-1} L^k \\
= \|x_1 - x_0\| L^n \sum_{k=0}^{m-n-1} L^k = \frac{L^n(1 - L^{m-n})}{1 - L} \|x_1 - x_0\| \leq \frac{L^n - L^m}{1 - L} \|x_1 - x_0\|
\]
for all \( m \geq n \geq 1 \). Since \( L \in (0, 1) \), the sequence \( L^n \to 0 \), so \((L^n)\) is a Cauchy sequence. Hence for every \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that
\[
|L^n - L^m| < \frac{1 - L}{\|x_1 - x_0\|} \varepsilon
\]
for all \( m \geq n > n_0 \). From the above inequality, \( \|x_m - x_n\| < \varepsilon \) for all \( m \geq n > n_0 \), so \((x_n)\) is a Cauchy sequence.

(b) Show that there is a unique point in \( K^N \) such that \( x = f(x) \).

**Solution:** Since \((x_n)\) is a Cauchy sequence it is convergent (completeness of \( K^N \)). Hence \( x_n \to x \) for some \( x \in K^N \). By assumption on \( f \)
\[
\|f(x_n) - f(x)\| \leq L\|x_n - x\| \to 0,
\]
so \( f(x_n) \to f(x) \) as \( n \to \infty \). Therefore
\[
f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} x_n = x
\]
as claimed. Suppose that \( f(x) = x \) and \( f(y) = y \). Then by the assumption
\[
\|x - y\| = \|f(x) - f(y)\| \leq L\|x - y\|.
\]
Since \( L \in (0, 1) \) this is only possible if \( x = y \), whence the uniqueness of \( x \).