The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 4 (Week 5)

MATH2962: Real and Complex Analysis (Advanced) Semester 1, 2017

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Material covered

(1) Series and their convergence, comparison tests

(2) Cauchy sequences and convergence

Outcomes

This tutorial helps you to

(1) have solid foundations in the more formal aspects of analysis, including a knowledge of precise definitions, how to apply them and the ability to write simple proofs;

(2) be able to work with inequalities, limits and limit inferior/superior;

(3) know and be able to apply convergence tests for sequences and series;

Summary of essential material

Series are sequences formed by summing terms $a_k \in \mathbb{K}^N$. We denote series by

$$\sum_{k=0}^{\infty} a_k.$$

We call

- $s_n := \sum_{k=0}^{n}$ then *n-th partial sum* of the series.
- (s_n) the *sequence of partial sums*

We say that the series $\sum_{k=0}^{\infty} a_k$

- is *convergent* if the sequence of partial sums (s_n) converges;
- is *divergent* if the sequence of partial sums (s_n) diverges.

Consider the class of series with ***non-negative terms***, that is, $a_k \geq 0$ for all $n \in \mathbb{N}$. Then the sequence of partial sums (s_n) is increasing, and hence

(s_n) converges if and only if it is bounded.

That fact is the basis for the comparison tests:

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Comparison test. Suppose that there exists $m \in \mathbb{N}$ so that $0 \leq a_k \leq b_k$ for all $k \geq m$.

- If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges.
- If $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ diverges.

Limit comparison test. Suppose that there exists $m \in \mathbb{N}$ so that $0 \leq a_k$ and $0 < b_k$ for all $k \geq m$, and that

$$\limsup_{n\to\infty} \frac{a_n}{b_n} < \infty \qquad \text{or equivalently} \qquad \left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}} \text{ is bounded}$$

- If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges.
- If $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ diverges.

Questions to complete during the tutorial

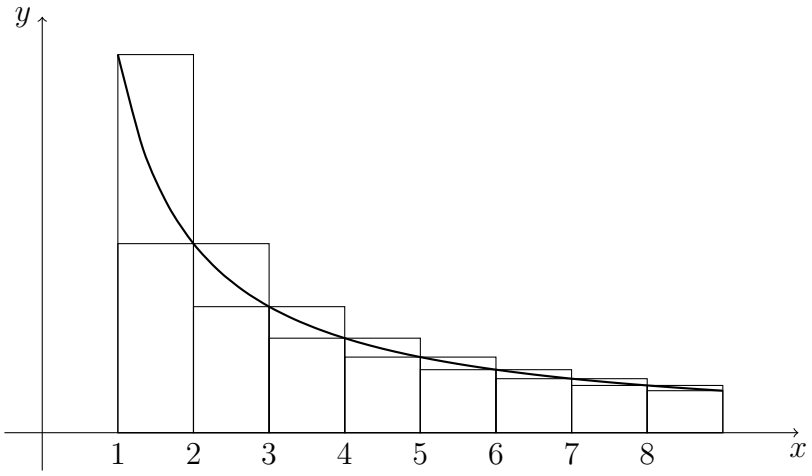
1. (a) Suppose that $f\colon [1,\infty) \to \mathbb{R}$ is a positive decreasing function and set $a_n := f(n)$ for $n \in \mathbb{N}$. For $n \geq 1$ define

$$s_n := \sum_{k=1}^{n} a_k \qquad \text{and} \qquad I_n := \int_1^n f(x)\,dx.$$

(i) Use upper and lower Riemann sums for the integral I_n to show that, for all $n \geq 2$,

$$s_n - a_1 \leq I_n \leq s_n - a_n.$$

Solution: Let $n \geq 2$. Then the integrals I_n exist as f is monotone. We partition the interval $[1, n]$ into $n-1$ sub-intervals of length 1. Because f is a decreasing function, the upper and lower Riemann sums for this partition are $a_1 + a_2 + \cdots + a_{n-1}$ and $a_2 + \cdots + a_n$, respectively. We illustrate that on the graph below.



Hence we get the inequality

$$s_n - a_1 = a_2 + \cdots + a_n \leq \int_1^n f(x)\,dx \leq a_1 + a_2 + \cdots + a_{n-1} = s_n - a_n$$

for all $n \geq 2$.

(ii) Hence establish the *Integral Test*: Suppose that $a_n = f(n)$ where $f(x)$ is positive decreasing on $[1, \infty)$. Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if

$$\int_1^{\infty} f(x)\,dx := \lim_{n\to\infty} \int_1^n f(x)\,dx < \infty.$$

Solution: Because f is assumed positive, I_n is an increasing sequence, and therefore convergent if and only if it is bounded. Similarly, the sequence of partial sums (s_n) is increasing, and therefore convergent if and only if it is bounded.
Assume first that (I_n) is bounded and therefore convergent. Hence assume that

$$I := \lim_{n\to\infty} I_n = \int_1^{\infty} f(x)\,dx < \infty$$

From the first inequality in the previous part

$$s_n \leq a_1 + I_n \leq a_1 + I < \infty$$

for all $n \geq 2$. Hence the sequence of partial sums s_n is bounded, so s_n converges.
Assume now that I_n is unbounded, then by the second inequality in the previous part

$$I_n \leq s_n$$

for all $n \geq 2$. Therefore (s_n) is unbounded as well, so diverges.

(b) Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

Solution: If $p \leq 0$, then $1/n^p \not\to 0$ as $n \to \infty$, and therefore the series diverges. If $p > 0$, then $1/x^p$ is decreasing to zero, and therefore convergence is possible. We distinguish some cases. If $p = 1$ then

$$I_n = \int_1^n \frac{1}{x^p}\,dx = \log x\Big|_1^n = \log n \xrightarrow{n\to\infty} \infty.$$

Hence I_n diverges. If $p \neq 1$, then

$$I_n = \int_1^n \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p}\bigg|_1^n = \frac{n^{1-p} - 1}{1-p}.$$

Hence (I_n) converges if $p > 1$, and diverges if $p \in (0, 1)$. From the Integral Test we conclude that (s_n) converges if and only if $p > 1$.

2. Determine which of the series below converge, and which diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+n+1}$;

Solution: Clearly $0 \leq \dfrac{1}{2n^2+n+1} \leq \dfrac{1}{2n^2}$. We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the original series converges by the comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2-n+1}$;

Solution: Note that for $\dfrac{n^2}{2n^2-n+1} \to \dfrac{1}{2}$ as $n \to \infty$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the original series converges by the limit comparison test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$;

Solution: Clearly $\dfrac{1}{2n-1} \geq \dfrac{1}{2n}$ for all $n \in \mathbb{N}$. As the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does the original series by the comparison test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+3\sqrt{n}}$;

Solution: Clearly, $\dfrac{1}{1+3\sqrt{n}} \geq \dfrac{1}{4\sqrt{n}} \geq \dfrac{1}{4n}$ for all $n \in \mathbb{N}$. As the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does the original series by the comparison test.

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n-1}{3^n+1}$;

Solution: Clearly $\dfrac{2^n-1}{3^n+1} \leq \left(\dfrac{2}{3}\right)^n$ for all $n \in \mathbb{N}$. Hence the series converges by comparison with the geometric series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n-1}$;

Solution: Note that $\dfrac{2^n+1}{3^n-1}\left(\dfrac{3}{2}\right)^n \to 1$. Since $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ converges, the original series converges by the limit comparison test.

(g) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$;

Solution: Clearly $\dfrac{1}{n^2 \log n} \leq \dfrac{1}{n^2}$ for $n \geq 3$, so the series converges by comparison to the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n}$;

Solution: Clearly $0 \leq \frac{1}{n} \leq \frac{\log n}{n}$ for $n \geq 3$. We know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so the original series diverges by the comparison test.

*(i) $\sum_{n=1}^{\infty} \frac{n}{4^n}$;

Solution: We know that exponential decay is stronger than polynomial growth. Hence we do a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}.$$

We have

$$\frac{\frac{n}{4^n}}{\frac{1}{2^n}} = n2^{-n} \to 0$$

as $n \to \infty$ by one of the elementary limits. Hence the series converges by the limit comparison test. We will discuss more efficient convergence tests for a sequence such as this one (root and ratio tests).

3. Consider two sequences (a_n) and (b_n) in $\mathbb{R}$ or $\mathbb{C}$ with $b_n \neq 0$ for all $n \in \mathbb{N}$. We call the sequence (a_n) equivalent to (b_n) if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

If that is the case, we write $a_n \sim b_n$.

(a) Show that equivalence of sequences is an equivalence relation, that is, it has the following properties:

(i) $a_n \sim a_n$ (reflexivity);

Solution: Clearly $\frac{a_n}{a_n} = 1 \to 1$, so $a_n \sim a_n$.

(ii) $a_n \sim b_n \Rightarrow b_n \sim a_n$ (symmetry);

Solution: By the limit laws $\frac{b_n}{a_n} = \frac{1}{\frac{a_n}{b_n}} \to \frac{1}{1} = 1$, so $b_n \sim a_n$ if $a_n \sim b_n$.

(iii) $a_n \sim b_n$ and $b_n \sim c_n$ implies that $a_n \sim c_n$ (transitivity).

Solution: By the limit laws $\frac{a_n}{c_n} = \frac{a_n}{b_n}\frac{b_n}{c_n} \to 1 \cdot 1 = 1$.

(b) Suppose that $a_n \sim b_n$. Show that (a_n) converges if and only if (b_n) converges. In case of convergence show that the limits of (a_n) and (b_n) are the same.

Solution: If $b_n \to b$, then by the limit laws $a_n = \frac{a_n}{b_n} b_n \to 1 \cdot b = b$. Hence, $a_n \to b$. If $a_n \sim b_n$, then by (a)(ii) we also have $b_n \sim a_n$. Hence, interchanging the roles of a_n and b_n in the above argument we see that $a_n \to a$ implies that $b_n \to a$.

Extra questions for further practice

4. Let $a_k, b_k > 0$ for all $k \in \mathbb{N}$ and suppose that $a_k \sim b_k$. Show that $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} b_k$ converges.

Solution: Since $\dfrac{a_n}{b_n} \to 1$, the convergence of $\sum_{k=0}^{\infty} b_k$ implies the convergence of $\sum_{k=0}^{\infty} a_k$ by the limit comparison test. Since also $\dfrac{b_n}{a_n} \to 1$, the convergence of $\sum_{k=0}^{\infty} a_k$ implies the convergence of $\sum_{k=0}^{\infty} b_k$, again by the limit comparison test, completing the proof of the statement.

5. Prove that the following sequences are equivalent and decide whether they converge or diverge.

(a) $\dfrac{4n+1}{3n-1} \sim \dfrac{4}{3}$

Solution: We have $\dfrac{4n+1}{3n-1}\dfrac{3}{4} = \dfrac{4+1/n}{3-1/n}\dfrac{3}{4} \to 1$. Since the constant sequence $3/4$ converges, the given sequence converges to $3/4$.

(b) $\dfrac{3^n+2^n}{3^n-2^n} \sim 1$

Solution: We have $\dfrac{3^n+2^n}{3^n-2^n} = \dfrac{1+(2/3)^n}{1-(2/3)^n} \to 1$. Since the constant sequence 1 converges, the given sequence converges to 1.

(c) $\dfrac{\sqrt{n^2+3^n}}{n^4+3n+1} \sim \dfrac{3^{n/2}}{n^4}$

Solution: We have $\dfrac{\sqrt{n^2+3^n}}{n^4+3n+1}\dfrac{n^4}{3^{n/2}} = \dfrac{\sqrt{n^2 3^{-n}+1}}{1+3n^{-3}+n^{-4}} \to 1$. Since the sequence $3^{n/2}/n^4$ diverges, the given sequence diverges.

*(d) $\ln n \sim s_n := \displaystyle\sum_{k=1}^{n} \frac{1}{k}$

Solution: For $n \geq 2$ we can write

$$\ln n = \int_1^n \frac{1}{x}\,dx$$

Using the estimate in Question 1(a) we get

$$\begin{aligned} s_n - 1 = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} &\leq \ln n \\ &= \int_1^n \frac{1}{x}\,dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} = s_n - \frac{1}{n} \end{aligned}$$

for all $n \geq 2$. Hence,

$$1 - \frac{1}{s_n} \leq \frac{\ln n}{s_n} \leq 1 - \frac{1}{n s_n}$$

for all $n \geq 2$. Using that $s_n \to \infty$ (divergence of harmonic series) and the squeeze law, we deduce that

$$\lim_{n\to\infty} \frac{\ln n}{s_n} = 1.$$

Therefore $\ln n \sim s_n$ as claimed. This shows that the partial sums of the harmonic series grow like $\ln n$.

6. Let (x_n) be a sequence in $\mathbb{R}^N$ and (a_n) a sequence in $\mathbb{R}$ with $a_n \geq 0$ for all $n \in \mathbb{N}$. Assume that $\|x_{n+1} - x_n\| \leq a_n$ for all $n \in \mathbb{N}$, and that the series $\sum_{k=0}^{\infty} a_k$ converges. Show that (x_n) is convergent.

Hint: Rewrite the sequence as a telescoping sum and apply the Cauchy criterion.

Solution: We first note that

$$x_{n+1} - x_0 = (x_{n+1} - x_n) + (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - x_0) = \sum_{k=0}^{n} (x_{k+1} - x_k)$$

and hence

$$x_{n+1} = x_0 + \sum_{k=0}^{n} (x_{k+1} - x_k).$$

We have therefore written the sequence (x_n) as a series, and can therefore attempt to use the Cauchy criterion for series. We note that from the assumptions

$$\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{\infty} a_k < \infty$$

for all $n \geq 0$. As $\sum_{k=0}^{\infty} a_k$ is convergent the Cauchy criterion implies that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$0 \leq \sum_{k=n+1}^{m} a_k < \varepsilon$$

for all $m > n > n_0$. Hence, using the triangle inequality and the assumptions,

$$\Big\| \sum_{k=n+1}^{m} (x_{k+1} - x_k) \Big\| \leq \sum_{k=n+1}^{m} \|x_{k+1} - x_k\| \leq \sum_{k=n+1}^{m} a_k < \varepsilon$$

for all $m > n > n_0$. Therefore the series $\sum_{k=0}^{\infty} (x_{k+1} - x_k)$ satisfies the Cauchy criterion and hence converges. We therefore conclude that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x_0 + \sum_{k=0}^{\infty} (x_{k+1} - x_k)$$

exists.

Challenge questions (optional)

***7.** Suppose that $f\colon \mathbb{K}^N \to \mathbb{K}^N$ is a function such that there exists $L \in (0, 1)$ with

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

for all $x, y \in \mathbb{K}^N$. A function with that property is called a *contraction* since any pair of image points is closer together than the original points.

(a) Let $x_0 \in \mathbb{K}^N$ and define $x_{n+1} := f(x_n)$ for $n \geq 0$. Show that (x_n) is a Cauchy sequence.

Solution: First consider the increments $\|x_{n+1} - x_n\|$. From the definition of the sequence and the property of f

$$\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1})\| \leq L\|x_n - x_{n-1}\| \tag{1}$$

for all $n \geq 1$. If we iterate this estimate starting from $n = 1$ we expect that

$$\|x_{n+1} - x_n\| \leq L^n \|x_1 - x_0\| \tag{2}$$

for all $n \in \mathbb{N}$. We give a proper proof by induction. For $n = 1$ the statement is obvious from (1) with $n = 1$. Suppose now the statement is true for some $n \geq 1$. Then by (1) and the induction hypothesis

$$\|x_{n+2} - x_{n+1}\| \leq L\|x_{n+1} - x_n\| \leq LL^n \|x_1 - x_0\| = L^{n+1}\|x_1 - x_0\|,$$

so the statement is true for $n + 1$. We use (2) to show that (x_n) is a Cauchy sequence. In particular, using the formula for the partial sum of a geometric series

$$\begin{aligned}\|x_m - x_n\| &\leq \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| \leq \|x_1 - x_0\| \sum_{k=n}^{m-1} L^k \\ &= \|x_1 - x_0\| L^n \sum_{k=0}^{m-n-1} L^k = \frac{L^n(1 - L^{m-n})}{1 - L}\|x_1 - x_0\| \leq \frac{L^n - L^m}{1 - L}\|x_1 - x_0\|\end{aligned}$$

for all $m \geq n \geq 1$. Since $L \in (0, 1)$, the sequence $L^n \to 0$, so (L^n) is a Cauchy sequence. Hence for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|L^n - L^m| < \frac{1 - L}{\|x_1 - x_0\|}\varepsilon$$

for all $m \geq n > n_0$. From the above inequality, $\|x_m - x_n\| < \varepsilon$ for all $m \geq n > n_0$, so (x_n) is a Cauchy sequence.

(b) Show that there is a unique point in $\mathbb{K}^N$ such that $x = f(x)$.

Solution: Since (x_n) is a Cauchy sequence it is convergent (completeness of $\mathbb{K}^N$). Hence $x_n \to x$ for some $x \in \mathbb{K}^N$. By assumption on f

$$\|f(x_n) - f(x)\| \leq L\|x_n - x\| \to 0,$$

so $f(x_n) \to f(x)$ as $n \to \infty$. Therefore

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$$

as claimed. Suppose that $f(x) = x$ and $f(y) = y$. Then by the assumption

$$\|x - y\| = \|f(x) - f(y)\| \leq L\|x - y\|.$$

Since $L \in (0, 1)$ this is only possible if $x = y$, whence the uniqueness of x.