The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 4 (Week 5)

MATH2962: Real and Complex Analysis (Advanced) — Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. (a) Suppose that $f\colon [1,\infty) \to \mathbb{R}$ is a positive decreasing function. Show that for every $n \geq 2$

$$f(2) + \cdots + f(n) \leq \int_1^n f(x)\,dx \leq f(1) + f(2) + \cdots + f(n-1). \tag{1}$$

Solution: The integrals on $[1,n]$ exist as f is monotone. We partition the interval $[1,n]$ into $(n-1)$ intervals of length 1. Because f is a decreasing function, the lower and upper Riemann sums for this partition are $f(2)+\cdots+f(n)$ and $f(1)+f(2)+\cdots+f(n-1)$, respectively. Hence we get the inequality (1).

(b) Assume that $f(x)$ is a positive decreasing function on $[1,\infty)$. Using (a), establish the following *Integral Test*: The series $\displaystyle\sum_{n=1}^{\infty} f(n)$ is convergent if and only if

$$\int_1^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_1^n f(x)\,dx < \infty. \tag{2}$$

Solution: We set $I_n = \int_1^n f(x)\,dx$ for $n \in \mathbb{N}\setminus\{0\}$. Let s_n denote the sequence of partial sums for $\displaystyle\sum_{n=1}^{\infty} f(n)$, that is $s_n = \sum_{k=1}^n f(k)$ for $n \geq 1$. Because f is assumed positive on $[1,\infty)$, we find that I_n and s_n are increasing sequences. Hence I_n converges as $n \to \infty$ if and only if I_n is bounded from above. Similarly, the series $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges if and only if s_n is bounded from above. We now prove the Integral Test.

"$\Rightarrow$" We show that if the series $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges, then I_n is bounded from above. The second inequality in (1) gives that $I_n \leq s_{n-1}$ for every $n \geq 2$. Since $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges, we have that s_n is convergent and thus bounded. Hence, I_n is bounded from above.

"$\Leftarrow$" We now assume that (2) holds and prove that s_n is bounded from above. From the first inequality in (1), we have $s_n \leq I_n + f(1)$ for every $n \geq 2$. Since I_n is convergent and thus bounded, we conclude that s_n is bounded from above.

(c) Use the Integral Test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

Solution: Notice that if $p \leq 0$, then $1/n^p$ does not converge to 0 as $n \to \infty$ so that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges. We now assume that $p > 0$ in which case $f(x) = 1/x^p$ is decreasing on $[1,\infty)$. We know that

$$I_n = \int_1^n \frac{1}{x^p}\,dx = \begin{cases} \left.\dfrac{x^{1-p}}{1-p}\right|_1^n = \dfrac{n^{1-p}-1}{1-p} & \text{if } p \neq 1, \\ \left.\log x\right|_1^n = \log n & \text{if } p = 1. \end{cases}$$

Hence I_n converges if and only if $p > 1$.

2. Determine which of the series below converge, and which diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+n+1}$;

Solution: Clearly $0 \leq \dfrac{1}{2n^2+n+1} \leq \dfrac{1}{2n^2}$. We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. Thus the original series converges by the *comparison test.*

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2-n+1}$;

Solution: Note that $\dfrac{n^2}{2n^2-n+1} \to \dfrac{1}{2}$ as $n \to \infty$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, we find that the original series converges by the *limit comparison test.*

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$;

Solution: Clearly $\dfrac{1}{2n-1} \geq \dfrac{1}{2n}$ for all $n \in \mathbb{N}$. As the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so does the original series by the *comparison test.*

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+3\sqrt{n}}$;

Solution: Clearly, $\dfrac{1}{1+3\sqrt{n}} \geq \dfrac{1}{4\sqrt{n}} \geq \dfrac{1}{4n}$ for all $n \in \mathbb{N}$. As the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so does the original series by the *comparison test.*

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n-1}{3^n+1}$;

Solution: Clearly $\dfrac{2^n-1}{3^n+1} \leq \left(\dfrac{2}{3}\right)^n$ for all $n \in \mathbb{N}$. Hence the series converges by comparison with the geometric series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n-1}$;

Solution: Note that $\dfrac{2^n+1}{3^n-1}\left(\dfrac{3}{2}\right)^n \to 1$. Since $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ converges, the original series converges by the *limit comparison test.*

(g) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2\log n}$;

Solution: Clearly $\frac{1}{n^2 \log n} \leq \frac{1}{n^2}$ for $n \geq 3$, so the series converges by comparison to the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(h) $\sum_{n=1}^{\infty} \frac{\log n}{n}$;

Solution: Clearly $0 \leq \frac{1}{n} \leq \frac{\log n}{n}$ for $n \geq 3$. We know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so the original series diverges by the *comparison test.*

3. Consider two sequences (a_n) and (b_n) in $\mathbb{R}$ or $\mathbb{C}$ with $b_n \neq 0$ for all $n \in \mathbb{N}$. We call the sequence (a_n) equivalent to (b_n) if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

If that is the case, we write $a_n \sim b_n$.

(a) Show that equivalence of sequences is an equivalence relation, that is, it has the following properties:

(i) $a_n \sim a_n$ (reflexivity);

Solution: Clearly $\frac{a_n}{a_n} = 1 \to 1$, so $a_n \sim a_n$.

(ii) $a_n \sim b_n \Rightarrow b_n \sim a_n$ (symmetry);

Solution: By the limit laws $\frac{b_n}{a_n} = \frac{1}{\frac{a_n}{b_n}} \to \frac{1}{1} = 1$, so $b_n \sim a_n$ if $a_n \sim b_n$.

(iii) $a_n \sim b_n$ and $b_n \sim c_n$ implies that $a_n \sim c_n$ (transitivity).

Solution: By the limit laws $\frac{a_n}{c_n} = \frac{a_n}{b_n}\frac{b_n}{c_n} \to 1 \cdot 1 = 1$.

(b) Suppose that $a_n \sim b_n$. Show that (a_n) converges if and only if (b_n) converges. In case of convergence show that the limits of (a_n) and (b_n) are the same.

Solution: If $b_n \to b$, then by the limit laws $a_n = \frac{a_n}{b_n} b_n \to 1 \cdot b = b$, so $a_n \to b$. From $a_n \sim b_n$ and (a)(ii), we also have $b_n \sim a_n$. Hence, the above shows that when $a_n \to a$ then $b_n \to a$.

Extra questions for further practice

4. Let $a_k, b_k > 0$ for all $k \in \mathbb{N}$ and suppose that $a_k \sim b_k$. Show that $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} b_k$ converges.

Solution: Since $\frac{a_n}{b_n} \to 1$, the convergence of $\sum_{k=0}^{\infty} b_k$ implies the convergence of $\sum_{k=0}^{\infty} a_k$ by the limit comparison test. Since also $\frac{b_n}{a_n} \to 1$, the convergence of $\sum_{k=0}^{\infty} a_k$ implies the convergence of $\sum_{k=0}^{\infty} b_k$, again by the limit comparison test, completing the proof of the statement.

5. Prove that the following sequences are equivalent and decide whether they converge or diverge.

(a) $\frac{4n+1}{3n-1} \sim \frac{4}{3}$

Solution: We have $\frac{4n+1}{3n-1}\frac{3}{4} = \frac{4+1/n}{3-/n1}\frac{3}{4} \to 1$. Since the constant sequence $3/4$ converges, the given sequence converges to $3/4$.

(b) $\dfrac{3^n+2^n}{3^n-2^n} \sim 1$

Solution: We have $\dfrac{3^n+2^n}{3^n-2^n} = \dfrac{1+(2/3)^n}{1-(2/3)^n} \to 1$. Since the constant sequence 1 converges, the given sequence converges to 1.

(c) $\dfrac{\sqrt{n^2+3^n}}{n^4+3n+1} \sim \dfrac{3^{n/2}}{n^4}$

Solution: We have $\dfrac{\sqrt{n^2+3^n}}{n^4+3n+1}\dfrac{n^4}{3^{n/2}} = \dfrac{\sqrt{n^2 3^{-n}+1}}{1+3n^{-3}+n^{-4}} \to 1$. Since the sequence $3^{n/2}/n^4$ diverges, the given sequence diverges.

*(d) $\ln n \sim s_n := \displaystyle\sum_{k=1}^{n} \frac{1}{k}$

Solution: Let $n \geq 2$ be a natural number. Since the harmonic series diverges, we have $s_n \to \infty$, where $s_n = \sum_{k=1}^n 1/k$. We know that

$$\ln n = \int_1^n \frac{1}{x}\,dx.$$

Using the estimate in Question 1(a), for every $n \geq 2$, we obtain that

$$s_n - 1 = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \ln n = \int_1^n \frac{1}{x}\,dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} = s_n - \frac{1}{n}.$$

Hence, we have that

$$1 - \frac{1}{s_n} \leq \frac{\ln n}{s_n} \leq 1 - \frac{1}{n s_n} \quad \text{for every } n \geq 2. \tag{3}$$

By passing to the limit $n \to \infty$ in (3) and using the squeeze law, we arrive at

$$\lim_{n\to\infty} \frac{\ln n}{s_n} = 1.$$

Therefore $\ln n \sim s_n$ as claimed.

Challenge questions (optional)

***6.** Suppose that $f\colon \mathbb{K}^N \to \mathbb{K}^N$ is a function such that there exists $L \in (0,1)$ with

$$\|f(x) - f(y)\| \leq L\|x-y\| \quad \text{for all } x, y \in \mathbb{K}^N. \tag{4}$$

(a) Given $x_0 \in \mathbb{K}^N$ and $x_{n+1} := f(x_n)$ for any $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.

Solution: Notice first that if $x_1 = x_0$, that is $f(x_0) = x_0$, then $x_n = x_0$ for every $n \geq 1$. Hence, x_n is trivially a Cauchy sequence. We can thus assume that $x_1 \neq x_0$. By definition, (x_n) is a Cauchy sequence if for every $\varepsilon > 0$, there exists $n_\varepsilon \geq 1$ such that

$$\|x_m - x_n\| < \varepsilon \quad \text{for every } m, n > n_\varepsilon. \tag{5}$$

We need only prove (5) for $m > n > n_\varepsilon$ since (5) with $m = n$ is obvious and the case $n > m > n_\varepsilon$ would follow by interchanging m and n (use also that $\|x_m - x_n\| = \|x_n - x_m\|$). To prove (5), we first need to get *an upper bound estimate* for $\|x_{n+1} - x_n\|$ for any $n \geq 1$. When $n = 1$, we have

$$\|x_2 - x_1\| = \|f(x_1) - f(x_0)\| \leq L\|x_1 - x_0\|,$$

where the inequality is true in view of our assumption (4). Similarly, we have

$$\|x_3 - x_2\| = \|f(x_2) - f(x_1)\| \leq L\|x_2 - x_1\| \leq L^2\|x_1 - x_0\|.$$

This suggests the following inequality

$$\|x_{n+1} - x_n\| \leq L^n\|x_1 - x_0\| \quad \text{for all } n \geq 1, \tag{6}$$

which we shall prove it by induction after n. For $n = 1$ the statement has already been proved. Suppose now the statement is true for $n \geq 1$. Then using the assumption (4) and the induction hypothesis, we find that

$$\|x_{n+2} - x_{n+1}\| = \|f(x_{n+1}) - f(x_n)\| \leq L\|x_{n+1} - x_n\| \leq LL^n\|x_1 - x_0\| = L^{n+1}\|x_1 - x_0\|,$$

so the statement (6) is also true for $n + 1$.

We use (6) and the triangle inequality to obtain *an upper bound estimate* for $\|x_m - x_n\|$ with $m > n \geq 1$. Indeed, we find that

$$\|x_m - x_n\| \leq \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| \leq \|x_1 - x_0\| \sum_{k=n}^{m-1} L^k = \|x_1 - x_0\| L^n \sum_{k=0}^{m-n-1} L^k. \tag{7}$$

The formula for the partial sum of a geometric series gives that

$$\sum_{k=0}^{m-n-1} L^k = \frac{1 - L^{m-n}}{1 - L}.$$

This, jointly with (7), implies that

$$\|x_m - x_n\| \leq \frac{L^n(1 - L^{m-n})}{1 - L}\|x_1 - x_0\| = \frac{L^n - L^m}{1 - L}\|x_1 - x_0\| \tag{8}$$

for all $m > n \geq 1$. Since $L \in (0, 1)$, the sequence $L^n \to 0$, so (L^n) is a Cauchy sequence. Hence for every $\varepsilon > 0$, there exists $n_\varepsilon \geq 1$ such that

$$|L^n - L^m| < \frac{1 - L}{\|x_1 - x_0\|}\varepsilon \quad \text{for all } m > n > n_\varepsilon.$$

From the above inequality and (8), we conclude that

$$\|x_m - x_n\| < \epsilon \quad \text{for every } m > n > n_\varepsilon.$$

Therefore, (5) is valid, which proves that (x_n) is a Cauchy sequence.

(b) Show that there is a unique point in $\mathbb{K}^N$ such that $x = f(x)$.

Solution: Since (x_n) is a Cauchy sequence it is convergent (by the completeness of $\mathbb{K}^N$). Hence $x_n \to x$ for some $x \in \mathbb{K}^N$. Our assumption (4) gives that

$$\|f(x_n) - f(x)\| \leq L\|x_n - x\| \to 0 \quad \text{as } n \to \infty.$$

Therefore, we have $f(x_n) \to f(x)$ as $n \to \infty$. It follows that

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$$

as claimed. Suppose that $f(x) = x$ and $f(y) = y$. Then by the assumption (4), we have

$$\|x - y\| = \|f(x) - f(y)\| \leq L\|x - y\|.$$

Since $L \in (0, 1)$ this is only possible if $x = y$, which proves the uniqueness of x.