The University of Sydney
School of Mathematics and Statistics

Tutorial 5 (Week 6)

MATH2962: Real and Complex Analysis (Advanced) — Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Classify the given series as either absolutely convergent, convergent, or divergent.

(a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n}{3n^2+1}$; (c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\sqrt{\frac{2^n}{1+4^n}}$; (e) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n^3}{2^n}$;

(b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\sqrt{n}}{3n^2+1}$; (d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n!}{n^n}$; (f) $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{1}{(\log n)^n}$.

2. Use the Cauchy condensation test to determine for which $p > 0$ the following series converge.

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}$; (b) $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n[\log(\log n)]^p\log n}$; (c) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\log n)^p}$

3. Let $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ be sequences in $\mathbb{C}$ and let $s_n := \sum_{j=0}^{n} a_j$ for $n \in \mathbb{N}$.

(a) Show that
$$\sum_{j=0}^{n} a_j b_j = b_{n+1}s_n - \sum_{k=0}^{n} s_k(b_{k+1}-b_k). \tag{1}$$

Hint: Use the definition of s_k and write the sum on the right hand side as a double sum. Then interchange the order of summation.

(b) Suppose that $(s_n)_{n\geq 0}$ is bounded, the series $\sum_{k=0}^{\infty}(b_{k+1}-b_k)$ converges absolutely, and $b_n \to 0$ as $n \to \infty$. Prove that the series $\sum_{j=0}^{\infty} a_j b_j$ converges.

(c) How does the above generalise the Leibniz criterion for alternating series?

Extra questions for further practice

4. Let $a > 0$. By using partial fractions, evaluate the following series:

(a) $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(a+k)(a+k+1)}$; *(b) $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(a+k)(a+k+1)(a+k+2)}$.

5. The purpose of this question is to show that if the terms change sign, the limit comparison test does not necessarily work.

(a) Show that the series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{k}}$ converges.

(b) Show that the series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sqrt{k}-(-1)^k}{k}$ diverges.

(c) Set $a_k := (-1)^{k+1}\dfrac{1}{\sqrt{k}}$ and $b_k := (-1)^{k+1}\dfrac{\sqrt{k}-(-1)^k}{k}$ for $k \geq 1$. Show that b_k/a_k converges as $k \to \infty$, but the limit comparison test does not apply.

Challenge questions (optional)

6. Let $A = [a_{ij}] \in \mathbb{K}^{N\times N}$ be an $N \times N$ matrix. Define a matrix norm by

$$\|A\| := \Big(\sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|^2\Big)^{1/2}.$$

Note that this is simply the usual norm in $\mathbb{K}^{N\times N} = \mathbb{K}^{N^2}$, and hence has all its properties including the triangle inequality. Prove that the matrix exponential

$$e^A := \sum_{k=0}^{\infty}\frac{1}{k!}A^k$$

converges absolutely. *Hint:* Use that $\|A^k\| \leq \|A\|^k$ by Tutorial 1, Question 5.