THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial 5 (Week 6)

MATH2962: Real and Complex Analysis (Advanced) Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Classify the given series as either absolutely convergent, convergent, or divergent.

(a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n}{3n^2+1}$;

Solution: The given series is of the form $\sum_{n=0}^{\infty}(-1)^n a_n$, where $a_n := \dfrac{n}{3n^2+1}$ for $n \in \mathbb{N}$. This is an alternating series, where $a_n \to 0$ as $n \to \infty$. Indeed, by the squeeze law, we have $0 \le \dfrac{n}{3n^2+1} \le \dfrac{1}{3n} \to 0$ as $n \to \infty$. At this moment, we cannot rule out convergence or absolute convergence of our series. We first check for absolute convergence. We have

$$a_n = \frac{n}{3n^2+1} \ge \frac{n}{4n^2} = \frac{1}{4n} \quad \text{for every } n \ge 1.$$

Since the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, by the comparison test, we conclude that $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges. Hence, the alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ is not absolutely convergent. To study the convergence of this series, we want to see if the Leibniz test applies. We check whether a_n is decreasing, at least for all $n \in \mathbf{N}$ sufficiently large. We have

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3(n+1)^2+1}\,\frac{3n^2+1}{n} \le \frac{3n^2+1}{3n(n+1)} \le \frac{3n^2+1}{3n^2+1} = 1$$

for all $n \ge 1$. Hence $(a_n)_{n\ge 1}$ is a positive sequence which decreases to 0 as $n \to \infty$. Thus by the Leibniz test, the initial series converges (conditionally).

(b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\sqrt{n}}{3n^2+1}$;

Solution: Note that $\dfrac{\sqrt{n}}{3n^2+1}\cdot\dfrac{n^{3/2}}{1} \to \dfrac{1}{3}$ as $n \to \infty$. Since the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is convergent (as a p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ with $p = 3/2 > 1$), it follows from the limit comparison test that the given series is absolutely convergent (and thus, also convergent).

(c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\sqrt{\frac{2^n}{1+4^n}}$;

Solution: Note that $\sqrt{\dfrac{2^n}{1+4^n}} \le \left(\dfrac{1}{\sqrt{2}}\right)^n$ for any $n \in \mathbb{N}$. Hence, the initial series converges absolutely by comparison with the geometric series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$.

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n!}{n^n}$;

Solution: We test for absolute convergence of the given series by using the Ratio Test. We denote $a_n := (-1)^n \frac{n!}{n^n}$ for $n \geq 1$. We see that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{1}{(1+1/n)^n} \to \frac{1}{e} \quad \text{as } n \to \infty.$$

Recall the basic limit $(1+1/n)^n \to e$ as $n \to \infty$. Since $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, we can apply the Ratio Test to conclude that the given series is absolutely convergent.

(e) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{n^3}{2^n}$;

Solution: We define $a_n := (-1)^n \frac{n^3}{2^n}$ for every $n \in \mathbb{N}$. We test for absolute convergence of the series $\sum_{n=0}^{\infty} a_n$ using the Ratio Test. We have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^3}{2^{n+1}} \frac{2^n}{n^3} = \frac{(n+1)^3}{2n^3} \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Since $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, we conclude that the given series converges absolutely.

(f) $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{1}{(\log n)^n}$.

Solution: Because $0 \leq \frac{1}{(\log n)^n} \leq \frac{1}{2^n}$ for $n \geq e^2$, we find that the series $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{1}{(\log n)^n}$ converges absolutely by comparison with the positive convergent geometric series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$.

2. Use the Cauchy condensation test to determine for which $p > 0$ the following series converge.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$;

Solution: We denote $a_n := \frac{1}{n(\log n)^p}$ for every $n \geq 2$. Since $p > 0$, we have that $(a_n)_{n\geq 2}$ is a sequence of positive numbers decreasing to 0 as $n \to \infty$. By the Cauchy condensation test, we have that $\displaystyle\sum_{n=2}^{\infty} a_n$ converges if and only if $\displaystyle\sum_{n=2}^{\infty} 2^n a_{2^n}$ converges. We see that

$$\sum_{n=2}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} 2^n \frac{1}{2^n(\log 2^n)^p} = \sum_{n=2}^{\infty} \frac{1}{n^p(\log 2)^p} = \frac{1}{(\log 2)^p} \sum_{n=2}^{\infty} \frac{1}{n^p}.$$

From lectures, we know that the "p-series" $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. Hence, the given series converges if $p > 1$ and diverges if $p \in (0, 1]$.

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n[\log(\log n)]^p \log n}$;

Solution: We denote $a_n = \dfrac{1}{n[\log(\log n)]^p \log n}$ for every $n \geq 3$. Observe that $(a_n)_{n\geq 3}$ is a sequence of positive numbers decreasing to 0 as $n \to \infty$. We again use the Cauchy condensation test so that the convergence of the given series is equivalent to the convergence of the series $\displaystyle\sum_{n=3}^{\infty} 2^n a_{2^n}$. We have

$$a_{2^n} = \frac{1}{2^n\big[\log(\log 2^n)\big]^p \log(2^n)} = \frac{1}{2^n\big[\log(n\log 2)\big]^p n\log 2}.$$

It follows that

$$\sum_{n=3}^{\infty} 2^n a_{2^n} = \frac{1}{(\log 2)} \sum_{n=3}^{\infty} \frac{1}{n\big[\log(n\log 2)\big]^p}.$$

Since

$$\frac{n(\log n)^p}{n\big[\log(n\log 2)\big]^p} = \Big(\frac{\log n}{\log n + \log(\log 2)}\Big)^p = \Big(\frac{1}{1+\frac{\log(\log 2)}{\log n}}\Big)^p \to 1 \quad \text{as } n\to\infty,$$

we find that the given series $\sum_{n=3}^{\infty} a_n$ converges if and only if $\sum_{n=3}^{\infty} \dfrac{1}{n(\log n)^p}$ converges. Hence, by using part (a), the given series converges if $p > 1$ and diverges if $p \in (0,1]$.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$

Solution: We again use the Cauchy condensation test and observe that it is sufficient to prove the convergence properties of

$$\sum_{n=2}^{\infty} 2^n \frac{1}{(\log 2^n)^p} = \sum_{n=2}^{\infty} \frac{2^n}{n^p(\log 2)^p} = \frac{1}{(\log 2)^p}\sum_{n=2}^{\infty}\frac{2^n}{n^p},$$

which diverges for all $p > 0$ since $2^n n^{-p} \to \infty$ for all $p > 0$.

3. Let $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ be sequences in $\mathbb{C}$ and let $s_n := \sum_{j=0}^{n} a_j$ for $n \in \mathbb{N}$.

(a) Show that

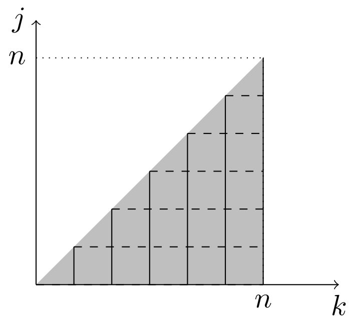
$$\sum_{j=0}^{n} a_j b_j = b_{n+1}s_n - \sum_{k=0}^{n} s_k(b_{k+1} - b_k). \tag{1}$$

Hint: Use the definition of s_k and write the sum on the right hand side as a double sum. Then interchange the order of summation.

Solution: By interchanging the order of summation, we obtain that

$$\sum_{k=0}^{n} s_k(b_{k+1}-b_k) = \sum_{k=0}^{n}\sum_{j=0}^{k} a_j(b_{k+1}-b_k) = \sum_{j=0}^{n}\sum_{k=j}^{n} a_j(b_{k+1}-b_k).$$

To find the limits for the summation it is useful to draw the region in the plane over which we take the sum:



The vertical lines represent summation over j first and then over k, the dashed horizontal lines the other way round. The above formula implies that

$$\begin{aligned}\sum_{k=0}^{n} s_k(b_{k+1}-b_k) &= \sum_{j=0}^{n}\sum_{k=j}^{n} a_j(b_{k+1}-b_k)\\ &= \sum_{j=0}^{n} a_j \sum_{k=j}^{n}(b_{k+1}-b_k) = \sum_{j=0}^{n} a_j(b_{n+1}-b_j) = b_{n+1}s_n - \sum_{j=0}^{n} a_j b_j.\end{aligned}$$

This proves our claim in (1).

(b) Suppose that $(s_n)_{n\geq 0}$ is bounded, the series $\sum_{k=0}^{\infty}(b_{k+1}-b_k)$ converges absolutely, and $b_n \to 0$ as $n \to \infty$. Prove that the series $\sum_{j=0}^{\infty} a_j b_j$ converges.

Solution: The idea is to use (1) and to show that the right-hand side of (1) converges as $n \to \infty$. Since $b_n \to 0$ as $n \to \infty$ and (s_n) is a bounded sequence, we obtain that $b_{n+1}s_n \to 0$ as $n \to \infty$. From the boundedness of (s_n), there exists a positive constant M such that $|s_k| \leq M$ for all $k \in \mathbb{N}$. Moreover, using that $\sum_{k=0}^{\infty}(b_{k+1}-b_k)$ converges absolutely, we deduce that

$$\sum_{k=0}^{\infty} |s_k||b_{k+1}-b_k| \leq M \sum_{k=0}^{\infty} |b_{k+1}-b_k| < \infty.$$

This shows that $\sum\limits_{k=0}^{\infty} s_k(b_{k+1}-b_k)$ is absolutely convergent and hence, convergent. Therefore, as $n \to \infty$, the right hand side of (1) converges. Consequently, the left hand side of (1) converges, which means that $\sum_{j=0}^{\infty} a_j b_j$ converges.

(c) How does the above generalise the Leibniz criterion for alternating series?

Solution: In the Leibniz criterion, we set $a_k := (-1)^k$. Then the sequence of partial sums $s_n = \sum_{k=0}^{n}(-1)^k$ is bounded. In fact, we have $|s_n| \leq 1$ for all $n \in \mathbb{N}$. Assuming that (b_n) decreases to zero, we obtain that

$$\sum_{k=0}^{n} |s_k(b_{k+1}-b_k)| \leq \sum_{k=0}^{n}(b_k - b_{k+1}) = b_0 - b_{n+1} \leq b_0$$

and hence the series $\sum_{k=0}^{\infty} s_k(b_{k+1}-b_k)$ converges absolutely. The above is more general since we do not require the series to alternate, but only need to require that (s_k) be bounded if (b_k) is decreasing to zero.

Extra questions for further practice

4. Let $a > 0$. By using partial fractions, evaluate the following series:

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(a+k)(a+k+1)}$;

Solution: Using partial fractions, the general term in the seris can be written as follows

$$\frac{1}{(a+k)(a+k+1)} = \frac{1}{a+k} - \frac{1}{a+k+1} \quad \text{for every } k \in \mathbb{N}.$$

Hence, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{k=0}^{n} \frac{1}{(a+k)(a+k+1)} &= \sum_{k=0}^{n}\Big(\frac{1}{a+k} - \frac{1}{a+k+1}\Big)\\ &= \frac{1}{a} - \frac{1}{a+1} + \frac{1}{a+1} - \frac{1}{a+2} + \cdots - \frac{1}{n+a} + \frac{1}{n+a} - \frac{1}{a+n+1}.\end{aligned}$$

Except the first and the last term, all terms cancel. Such a series is called a *telescoping series*. The sequence (s_n) of n-th partial sums is thus given by

$$s_n = \sum_{k=0}^{n} \frac{1}{(a+k)(a+k+1)} = \frac{1}{a} - \frac{1}{a+n+1} \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Letting $n \to \infty$, we conclude that

$$\sum_{k=0}^{\infty} \frac{1}{(a+k)(a+k+1)} = \lim_{n\to\infty} s_n = \frac{1}{a}.$$

*(b) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(a+k)(a+k+1)(a+k+2)}$.

Solution: Note that the given non-negative series is convergent by the comparison test with the convergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$. But the comparison test does not give us the limit of the series. Using partial fractions, we can write the general term in the series as follows

$$\begin{aligned}\frac{1}{(a+k)(a+k+1)(a+k+2)} &= \frac{1}{2(a+k+1)}\Big(\frac{1}{a+k} - \frac{1}{a+k+2}\Big) \\ &= \frac{1}{2}\left(\frac{1}{(a+k)(a+k+1)} - \frac{1}{(a+k+1)(a+k+2)}\right).\end{aligned}$$

Hence, if t_n denotes the n-th partial sum of the given series, then

$$\begin{aligned} t_n := &\sum_{k=0}^{n} \frac{1}{(a+k)(a+k+1)(a+k+2)} \\ = &\sum_{k=0}^{n} \frac{1}{2}\left(\frac{1}{(a+k)(a+k+1)} - \frac{1}{(a+k+1)(a+k+2)}\right) \\ = &\frac{1}{2}\sum_{k=0}^{n} \frac{1}{(a+k)(a+k+1)} - \frac{1}{2}\sum_{k=0}^{n} \frac{1}{(a+k+1)(a+k+2)}.\end{aligned}$$

Now the two sums on the right hand side are partial sums as in (2). Hence,

$$\begin{aligned} t_n &= \frac{1}{2}\left(\frac{1}{a} - \frac{1}{a+n+1}\right) - \frac{1}{2}\left(\frac{1}{a+1} - \frac{1}{a+n+2}\right) \\ &= \frac{1}{2a(a+1)} - \frac{1}{2(a+n+1)(a+n+2)}\end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain that

$$\sum_{k=0}^{\infty} \frac{1}{(a+k)(a+k+1)(a+k+2)} = \lim_{n\to\infty} t_n = \frac{1}{2a(a+1)}.$$

Remark. We can, in fact, obtain a more general result: If $a > 0$ and $m \in \mathbb{N} \setminus \{0\}$, then

$$\sum_{k=0}^{\infty} \frac{1}{\Pi_{j=0}^{m}(a+k+j)} = \frac{1}{m\,\Pi_{j=0}^{m-1}(a+j)}. \tag{3}$$

Indeed, by induction, it can be shown that the n-th partial sum of this series is

$$\sum_{k=0}^{n} \frac{1}{\Pi_{j=0}^{m}(a+k+j)} = \frac{1}{m\,\Pi_{j=0}^{m-1}(a+j)} - \frac{1}{m\,\Pi_{j=1}^{m}(a+n+j)} \quad \text{for all } n \in \mathbb{N}.$$

Passing to the limit $n \to \infty$, we conclude the claim of (3).

5. The purpose of this question is to show that if the terms change sign, the limit comparison test does not necessarily work.

(a) Show that the series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{k}}$ converges.

Solution: The series converges by the Leibniz test since $1/\sqrt{k}$ decreases to zero.

(b) Show that the series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sqrt{k}-(-1)^k}{k}$ diverges.

Solution: Note that

$$\sum_{k=1}^{n}(-1)^{k+1}\frac{\sqrt{k}-(-1)^k}{k}=\sum_{k=1}^{n}\Big((-1)^{k+1}\frac{1}{\sqrt{k}}+\frac{1}{k}\Big)=\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{\sqrt{k}}+\sum_{k=1}^{n}\frac{1}{k}.$$

The first of the partial sums converges by the Leibniz test, but the second is the harmonic series which diverges to infinity. Hence the original series diverges to infinity.

(c) Set $a_k := (-1)^{k+1}\dfrac{1}{\sqrt{k}}$ and $b_k := (-1)^{k+1}\dfrac{\sqrt{k}-(-1)^k}{k}$ for $k \geq 1$. Show that b_k/a_k converges as $k \to \infty$, but the limit comparison test does not apply.

Solution: We have

$$\frac{b_k}{a_k}=\frac{\sqrt{k}-(-1)^k}{k}\cdot\frac{\sqrt{k}}{1}=1-\frac{(-1)^k}{\sqrt{k}}\to 1 \quad\text{as } k\to\infty.$$

However, as shown above, the convergence of $\sum_{k=1}^{\infty} a_k$ does not imply the convergence of $\sum_{k=1}^{\infty} b_k$. Hence the limit comparison test does not necessarily work if the terms have both signs. (It works if the series are absolutely convergent.)

Challenge questions (optional)

6. Let $A = [a_{ij}] \in \mathbb{K}^{N\times N}$ be an $N \times N$ matrix. Define a matrix norm by

$$\|A\| := \Big(\sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|^2\Big)^{1/2}.$$

Note that this is simply the usual norm in $\mathbb{K}^{N\times N} = \mathbb{K}^{N^2}$, and hence has all its properties including the triangle inequality. Prove that the matrix exponential

$$e^A := \sum_{k=0}^{\infty}\frac{1}{k!}A^k$$

converges absolutely. *Hint:* Use that $\|A^k\| \leq \|A\|^k$ by Tutorial 1, Question 5.

Solution: By the hint we have

$$\Big\|\frac{A^k}{k!}\Big\| \leq \frac{\|A\|^k}{k!}$$

for all $k \geq 1$. Since

$$\exp(\|A\|)=\sum_{k=0}^{\infty}\frac{1}{k!}\|A\|^k$$

converges, it follows that the matrix exponential converges absolutely for every matrix A.