THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 6 (Week 7)

MATH2962: Real and Complex Analysis (Advanced) — Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Consider the power series $\sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{K}^N$ and suppose that

$$\alpha = \lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} \text{ exists in } [0,\infty]. \tag{1}$$

Show that the radius of convergence ρ for the given power series can be computed as $\rho = 1/\alpha$.

2. Determine the radius of convergence ϱ of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{z^n}{n^p}$, $p > 0$;

(b) $\sum_{n=0}^{\infty} \frac{n^2}{2^n} z^{3n}$;

(c) $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$;

*(d) $\sum_{n=1}^{\infty} (\log n) z^n$;

(e) $\sum_{n=1}^{\infty} n^{1/n} z^n$;

*(f) $1 - \frac{z}{2} + \frac{z^2}{3^2} - \frac{z^3}{2^3} + \frac{z^4}{3^4} - \frac{z^5}{2^5} + \cdots$.

Extra questions for further practice

3. Find power series expansions in z and their radius of convergence for the following maps.
Hint: Use the formula for the geometric series

$$\frac{1}{a-z} = \frac{1}{a}\frac{1}{(1-z/a)} = \frac{1}{a}\sum_{k=0}^{\infty}\left(\frac{z}{a}\right)^k, \tag{GS}$$

which converges if and only if $|z/a| < 1$, that is, $|z| < |a|$. Also make use of Cauchy products.

(a) $\frac{1}{3-z}$; (b) $\frac{1}{(3-z)^2}$; *(c) $\frac{1}{z^2-5z+6}$; *(d) $\frac{z+1}{z-1}$.

4. Consider the series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ with $a_k = b_k = (-1)^k (k+1)^{-1/2}$. The *Cauchy product* of the two series is $\sum_{n=0}^{\infty} c_n$, where c_n is defined by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad \text{for } n \in \mathbb{N}.$$

Prove that the Cauchy product diverges.

Hint: Use that $\sqrt{ab} \le (a+b)/2$ for all $a, b \ge 0$ (Proof?) to estimate $|c_n|$ from below.

Remark. Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be two convergent series in $\mathbb{K}$ with sums A and B, respectively.

(a) If their Cauchy product $\sum_{n=0}^{\infty} c_n$ is also convergent, then it has the sum $C = AB$ (**Abel's Theorem**).

(b) If **at least one** of the two convergent series is **absolutely convergent**, then their Cauchy product is also **convergent** and has the sum $C = AB$ (**Mertens' Theorem**).

(c) If both series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are **absolutely convergent**, then from lectures we know that their Cauchy product $\sum_{n=0}^{\infty} c_n$ is **absolutely convergent** (**Cauchy's Theorem**).

(d) If both convergent series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are **not absolutely convergent**, then their Cauchy product **may diverge**. The above exercise illustrates this situation. Indeed, the series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ with $a_k = b_k = (-1)^k(k+1)^{-1/2}$ are convergent by the Leibniz test. However, they are not absolutely convergent since the series $\sum_{k=0}^{\infty}(k+1)^{-1/2} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is divergent as a p-series with $p = 1/2 < 1$.

Challenge questions (optional)

***5.** Let $B \in \mathbb{R}^{N\times N}$ be a matrix. Define $\|B\|$ be as in Tutorial 1, Question **5**.

(a) Show that the series

$$\sum_{k=0}^{\infty} B^k$$

converges in $\mathbb{R}^{N\times N}$ if

$$\mathrm{s}(B) := \limsup_{k\to\infty} \sqrt[k]{\|B^k\|} < 1,$$

and in particular if $\|B\| < 1$. (The series is called the *Neumann series* for the matrix B.)

(b) If the Neumann series $\sum_{k=0}^{\infty} B^k$ converges, show that $I - B$ is invertible, and that

$$(I-B)^{-1} = \sum_{k=0}^{\infty} B^k. \tag{2}$$

(Note the similarity to the geometric series.)

(c) Show that $|\lambda| \le \mathrm{s}(B)$ for all eigenvalues λ of B, where $\mathrm{s}(B)$ is as in (a). Give an example where equality holds.

(d) Show that the above series may converge, no matter how large $\|B\|$ is.

***6.** Try to construct an array x_{jk} $(j, k \in \mathbb{N})$, such that the row and the column series converge absolutely, but $\sum_{i=0}^{\infty} x_{\sigma(i)}$ diverges for some bijection $\sigma\colon \mathbb{N} \to \mathbb{N}\times\mathbb{N}$.