The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 6 (Week 7)

MATH2962: Real and Complex Analysis (Advanced) Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
Lecturer: Florica Cîrstea

Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Consider the power series $\sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{K}^N$ and suppose that

$$\alpha = \lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} \text{ exists in } [0,\infty]. \tag{1}$$

Show that the radius of convergence ρ for the given power series can be computed as $\rho = 1/\alpha$.

Solution: We define $b_n = a_n z^n$ for $n \in \mathbb{N}$. The given series is $\sum_{n=0}^{\infty} b_n$ for which we apply the ratio test. Hence, the series $\sum_{n=0}^{\infty} b_n$ converges absolutely if $\limsup_{n\to\infty} \frac{\|b_{n+1}\|}{\|b_n\|} < 1$ and diverges if $\liminf_{n\to\infty} \frac{\|b_{n+1}\|}{\|b_n\|} > 1$. Since we have

$$\frac{\|b_{n+1}\|}{\|b_n\|} = \frac{\|a_{n+1}\||z^{n+1}|}{\|a_n\||z^n|} = |z|\frac{\|a_{n+1}\|}{\|a_n\|},$$

it follows that the series $\sum_{n=0}^{\infty} b_n$ converges absolutely if $|z|\limsup\limits_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$ and diverges if $|z|\liminf\limits_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} > 1$. If the limit α in (1) exists, then the limit superior and limit inferior coincide with α. Thus the series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $|z| < 1/\alpha$ and diverges if $|z| > 1/\alpha$. So, the radius of convergence ρ for the power series $\sum_{n=0}^{\infty} a_n z^n$ is $\rho = 1/\alpha$.

Remark. The above provides an alternative formula to compute the radius of convergence, but only works if the limit in (1) exists. If that limit does *not* exist, then we either use the formula from the *Cauchy–Hadamard Theorem* based on the root test (see lectures) or other means.

2. Determine the radius of convergence ϱ of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{z^n}{n^p}$, $p > 0$;

Solution: We define $a_n = 1/n^p$ for $n \geq 1$. We have that

$$\alpha = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1.$$

Hence, by Question **1** the radius of convergence for the given series is 1 for all $p > 0$.

(b) $\sum_{n=0}^{\infty} \frac{n^2}{2^n} z^{3n}$;

Solution: This is a power series $\sum_{k=0}^{\infty} a_k z^k$, where a_k is given by

$$a_k := \begin{cases} \dfrac{n^2}{2^n} & \text{if } k = 3n \text{ and } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We find the radius of convergence using the Cauchy–Hadamard Theorem. When $k \in \mathbb{N}$ is not a multiple of 3, then $\sqrt[k]{|a_k|} = 0$. If $k \in \mathbb{N}$ is a multiple of 3, that is $k = 3n$, then

$$\sqrt[k]{|a_k|} = \sqrt[3n]{|a_{3n}|} = \sqrt[3n]{\frac{n^2}{2^n}} = \frac{(\sqrt[n]{n})^{2/3}}{2^{1/3}} \to \frac{1}{2^{1/3}} \quad \text{as } k = 3n \to \infty.$$

Hence, we obtain that $\limsup\limits_{k\to\infty} \sqrt[k]{|a_k|} = \dfrac{1}{2^{1/3}}$. Then, by the Cauchy–Hadamard Theorem, the radius of convergence is $\varrho = \dfrac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} = 2^{1/3}$.

Remark. Another approach is to denote $y = z^3$ and observe that the given series is a power series in y, namely $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n} y^n$. For the latter series (in y), we can find that its radius of convergence is 2 using the ratio test (or root test). Therefore, by the definition of radius of convergence, the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n} y^n$ converges when $|y| < 2$ and diverges when $|y| > 2$. Since $|z|^3 = |y|$, we obtain that the initial series $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n} z^{3n}$ converges for $|z| < 2^{1/3}$ and diverges when $|z| > 2^{1/3}$, proving that its radius of convergence is $2^{1/3}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$;

Solution: We denote $a_n = \dfrac{n^n}{n!}$ for $n \geq 1$ and $a_0 = 0$. We compute the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ using the ratio test in Question 1. We have

$$\alpha := \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Hence by Question 1, the radius of convergence is $\varrho = 1/\alpha = 1/e$.

*(d) $\displaystyle\sum_{n=1}^{\infty} (\log n) z^n$;

Solution: Let $a_n = \log n$ for $n \geq 1$. Again we can use the ratio test, since

$$\alpha := \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\log(n+1)}{\log n} = \lim_{n\to\infty} \frac{\log n + \log(1 + 1/n)}{\log n} = 1.$$

By Question 1, the radius of convergence is $1/\alpha = 1$.

Remark. We could also use the Cauchy–Hadamard Theorem. For $n \geq 3$, we have $1 \leq \sqrt[n]{\log n} \leq \sqrt[n]{n} \to 1$ as $n \to \infty$. Thus, by the squeeze law,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\log n} = 1.$$

Hence, the radius of convergence is $\varrho = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = 1$.

(e) $\displaystyle\sum_{n=1}^{\infty} n^{1/n} z^n$;

Solution: Let $a_n = n^{1/n}$ for $n \geq 1$ and $a_0 = 0$. We compute the radius of convergence for the power series $\sum_{n=0}^{\infty} a_n z^n$ using the Cauchy–Hadamard Theorem. Since $1 \leq a_n \leq n$ for $n \geq 1$, we obtain that

$$1 \leq \sqrt[n]{|a_n|} \leq \sqrt[n]{n} \quad \text{for } n \geq 1.$$

We know that $\sqrt[n]{n} \to 1$ as $n \to \infty$. Hence, by the squeeze law, we find that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1.$$

Therefore, the radius of convergence of the given series is $\varrho = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = 1.$

*(f) $1 - \dfrac{z}{2} + \dfrac{z^2}{3^2} - \dfrac{z^3}{2^3} + \dfrac{z^4}{3^4} - \dfrac{z^5}{2^5} + \cdots.$

Solution: This is a power series $\sum_{n=0}^{\infty} a_n z^n$, where a_n is given by $\dfrac{1}{3^n}$ if $n \in \mathbb{N}$ is even, and $-\dfrac{1}{2^n}$ if $n \in \mathbb{N}$ is odd. It follows that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{\dfrac{1}{3^n}} = \dfrac{1}{3} & \text{if } n \geq 1 \text{ is even,} \\ \sqrt[n]{\dfrac{1}{2^n}} = \dfrac{1}{2} & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

Hence, $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1/2$ since the limit superior is the largest accumulation point. By the Cauchy–Hadamard Theorem, we have

$$\varrho = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = 2.$$

Extra questions for further practice

3. Find power series expansions in z and their radius of convergence for the following maps.

Hint: Use the formula for the geometric series

$$\frac{1}{a-z} = \frac{1}{a}\frac{1}{(1-z/a)} = \frac{1}{a}\sum_{k=0}^{\infty}\left(\frac{z}{a}\right)^k, \tag{GS}$$

which converges if and only if $|z/a| < 1$, that is, $|z| < |a|$. Also make use of Cauchy products.

(a) $\dfrac{1}{3-z}$;

Solution: By the above formula, we have

$$\frac{1}{3-z} = \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k = \sum_{k=0}^{\infty}\frac{z^k}{3^{k+1}}.$$

The radius of convergence is $\varrho = 3$.

(b) $\dfrac{1}{(3-z)^2}$;

Solution: Since the series $\dfrac{1}{3-z} = \sum_{k=0}^{\infty} a_k z^k$ with $a_k = \dfrac{1}{3^{k+1}}$ is absolutely convergent for $|z| < 3$, we can write $1/(3-z)^2$ as a Cauchy product. So, for $|z| < 3$ we have

$$\frac{1}{(3-z)^2} = \left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{k=0}^{\infty} a_k z^k\right) = \sum_{n=0}^{\infty} c_n z^n,$$

where c_n is given by

$$c_n = \sum_{k=0}^{n} a_k a_{n-k} = \sum_{k=0}^{n} \frac{1}{3^{k+1}}\frac{1}{3^{n-k+1}} = \sum_{k=0}^{n}\frac{1}{3^{n+2}} = \frac{n+1}{3^{n+2}}.$$

Thus, for $|z| < 3$ we find that

$$\frac{1}{(3-z)^2} = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} \frac{n+1}{3^{n+2}} z^n.$$

Since $\lim_{n\to\infty} \dfrac{|c_{n+1}|}{|c_n|} = \dfrac{1}{3}$, it follows that the radius of convergence for $\dfrac{1}{(3-z)^2}$ is 3.

*(c) $\dfrac{1}{z^2-5z+6}$;

Solution: We first factorize the denominator and then use partial fractions:

$$\frac{1}{z^2-5z+6} = \frac{1}{(z-2)(z-3)} = \frac{1}{2-z} - \frac{1}{3-z}.$$

Then by applying (GS) with $a = 2$ and $a = 3$, we obtain that for $|z| < 2$

$$\frac{1}{z^2-5z+6} = \frac{1}{2-z} - \frac{1}{3-z} = \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} - \sum_{k=0}^{\infty} \frac{z^k}{3^{k+1}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{k+1}} - \frac{1}{3^{k+1}} \right) z^k.$$

Since $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}}$ diverges for $|z| > 2$, the radius of convergence for $\dfrac{1}{z^2-5z+6}$ is 2.

*(d) $\dfrac{z+1}{z-1}$.

Solution: We observe that

$$\frac{z+1}{z-1} = \frac{(z-1)+2}{z-1} = 1 + \frac{2}{z-1} = 1 - \frac{2}{1-z}.$$

Hence, by using (GS), we find that

$$\frac{z+1}{z-1} = 1 - 2\sum_{k=0}^{\infty} z^k = -1 - 2z - 2z^2 - 2z^3 - \dots$$

if and only if $|z| < 1$. Hence, the radius of convergence for $(z+1)/(z-1)$ is $\varrho = 1$.

4. Consider the series $\displaystyle\sum_{k=0}^{\infty} a_k$ and $\displaystyle\sum_{k=0}^{\infty} b_k$ with $a_k = b_k = (-1)^k (k+1)^{-1/2}$. The *Cauchy product* of the two series is $\displaystyle\sum_{n=0}^{\infty} c_n$, where c_n is defined by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad \text{for } n \in \mathbb{N}.$$

Prove that the Cauchy product diverges.

Hint: Use that $\sqrt{ab} \le (a+b)/2$ for all $a, b \ge 0$ (Proof?) to estimate $|c_n|$ from below.

Remark. Let $\displaystyle\sum_{k=0}^{\infty} a_k$ and $\displaystyle\sum_{k=0}^{\infty} b_k$ be two convergent series in $\mathbb{K}$ with sums A and B, respectively.

(a) If their Cauchy product $\displaystyle\sum_{n=0}^{\infty} c_n$ is also convergent, then it has the sum $C = AB$ (**Abel's Theorem**).

(b) If **at least one** of the two convergent series is **absolutely convergent**, then their Cauchy product is also **convergent** and has the sum $C = AB$ (**Mertens' Theorem**).

(c) If both series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are **absolutely convergent**, then from lectures we know that their Cauchy product $\sum_{n=0}^{\infty} c_n$ is **absolutely convergent** (**Cauchy's Theorem**).

(d) If both convergent series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are **not absolutely convergent**, then their Cauchy product **may diverge**. The above exercise illustrates this situation. Indeed, the series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ with $a_k = b_k = (-1)^k(k+1)^{-1/2}$ are convergent by the Leibniz test. However, they are not absolutely convergent since the series $\sum_{k=0}^{\infty}(k+1)^{-1/2} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is divergent as a p-series with $p = 1/2 < 1$.

Solution: If we can prove that $c_n \not\to 0$ as $n \to \infty$, then we conclude that $\sum_{n=0}^{\infty} c_n$ diverges. We next prove that $|c_n| \not\to 0$ as $n \to \infty$ by establishing that $\liminf_{n\to\infty} |c_n| > 0$. By definition of c_n, we have

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n}(-1)^k(-1)^{n-k}\frac{1}{\sqrt{k+1}}\frac{1}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}.$$

We want to estimate $|c_n|$ from below. The arithmetic–geometric inequality gives that if $a, b \geq 0$, then $\sqrt{ab} \leq (a+b)/2$. By this inequality with $a = k+1$ and $b = n-k+1$, we have

$$\sqrt{(k+1)(n-k+1)} \leq \frac{(k+1)+(n-k+1)}{2} = \frac{n+2}{2} \quad \text{for } k = 0, 1, \dots, n.$$

Hence, for all $n \in \mathbb{N}$, we obtain a lower bound for $|c_n|$ as follows

$$|c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2} \geq 1.$$

This means that $\liminf_{n\to\infty} c_n \geq 1$ so that $|c_n| \not\to 0$ as $n \to \infty$. Hence, $c_n \not\to 0$ as $n \to \infty$, which proves that the Cauchy product $\sum_{n=0}^{\infty} c_n$ diverges.

Remark. To prove the arithmetic-geometric inequality, use that $0 \leq (a-b)^2 = a^2 + b^2 - 2ab$. By adding $4ab$ in both sides of the inequality, we obtain that

$$4ab \leq a^2 + b^2 + 2ab = (a+b)^2.$$

Dividing by 4 and taking square roots, we find that $\sqrt{ab} \leq (a+b)/2$ for any $a, b \geq 0$.

Challenge questions (optional)

***5.** Let $B \in \mathbb{R}^{N\times N}$ be a matrix. Define $\|B\|$ be as in Tutorial 1, Question **5**.

(a) Show that the series

$$\sum_{k=0}^{\infty} B^k$$

converges in $\mathbb{R}^{N\times N}$ if

$$\mathrm{s}(B) := \limsup_{k\to\infty} \sqrt[k]{\|B^k\|} < 1,$$

and in particular if $\|B\| < 1$. (The series is called the *Neumann series* for the matrix B.)

Solution: By the root test, the series $\sum_{k=0}^{\infty} B^k$ converges if $s(B) < 1$. Since $\|B^k\| \leq \|B\|^k$ for all $k \in \mathbb{N}$, we have $s(B) \leq \|B\|$. Hence, if $\|B\| < 1$, then the series $\sum_{k=0}^{\infty} B^k$ converges.

(b) If the Neumann series $\sum_{k=0}^{\infty} B^k$ converges, show that $I - B$ is invertible, and that

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k. \tag{2}$$

(Note the similarity to the geometric series.)

Solution: We set $S := \sum_{k=0}^{\infty} B^k$ and let $S_n := \sum_{k=0}^{n} B^k$ be the n-th partial sum. Then by the convergence hypothesis of the Neumann series, we have that

$$\lim_{n\to\infty} \|S_n - S\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|B^n\| = 0. \tag{3}$$

Let I denote the identity matrix in $\mathbb{R}^{N\times N}$. To prove (2), we need to show that

$$(I - B)S = S(I - B) = I. \tag{4}$$

To this end, we demonstrate that

$$\begin{cases} \|(I - B)S - I\| = \lim\limits_{n\to\infty} \|(I - B)S_n - I\| = 0, \\ \|S(I - B) - I\| = \lim\limits_{n\to\infty} \|S_n(I - B) - I\| = 0. \end{cases} \tag{5}$$

We first prove the second identities in (5). We notice that

$$(I - B)S_n - I = (I - B)\sum_{k=0}^{n} B^k - I = \sum_{k=0}^{n} B^k - \sum_{k=0}^{n} B^{k+1} - I = -B^{n+1}. \tag{6}$$

Similarly, we find that

$$S_n(I - B) - I = \left(\sum_{k=0}^{n} B^k\right)(I - B) - I = \sum_{k=0}^{n} B^k - \sum_{k=0}^{n} B^{k+1} - I = -B^{n+1}. \tag{7}$$

Using (3), (6) and (7), we obtain that

$$\|(I - B)S_n - I\| = \|S_n(I - B) - I\| = \|B^{n+1}\| \to 0 \text{ as } n \to \infty. \tag{8}$$

Hence, to conclude (5), it remains to show the first equalities in the relations of (5). To this aim, we prove that $(I - B)S_n \to (I - B)S$ and $S_n(I - B) \to S(I - B)$ as $n \to \infty$. Indeed, using (3), we infer that

$$\begin{cases} \|(I - B)S_n - (I - B)S\| = \|(I - B)(S_n - S)\| \leq \|I - B\|\|S_n - S\| \to 0 \quad \text{as } n \to \infty, \\ \|S_n(I - B) - S(I - B)\| = \|(S_n - S)(I - B)\| \leq \|S_n - S\|\|I - B\| \to 0 \quad \text{as } n \to \infty. \end{cases}$$

It thus follows that

$$(I - B)S_n - I \to (I - B)S - I \quad \text{and} \quad S_n(I - B) - I \to S(I - B) - I \text{ as } n \to \infty. \tag{9}$$

From (8) and (9), we conclude the assertions of (5). Hence, (4) holds.

(c) Show that $|\lambda| \leq \mathrm{s}(B)$ for all eigenvalues λ of B, where $\mathrm{s}(B)$ is as in (a). Give an example where equality holds.

Solution: We prove the contrapositive of the assertion, that is $|\lambda| > s(B)$ implies that λ is not an eigenvalue of B. To this aim, we assume that $|\lambda| > s(B)$. Since $\lambda \neq 0$, we can write $\lambda I - B = \lambda(I - \lambda^{-1}B)$. Hence, $\lambda I - B$ is invertible if and only if $I - \lambda^{-1}B$ is invertible. By part (a), the matrix $I - \lambda^{-1}B$ is invertible if

$$\mathrm{s}(\lambda^{-1}B) = \limsup_{k\to\infty} \|\lambda^{-k}B^k\|^{1/k} = |\lambda|^{-1} \limsup_{k\to\infty} \|B^k\|^{1/k} = |\lambda|^{-1}\,\mathrm{s}(B) < 1.$$

Hence, if $|\lambda| > \mathrm{s}(B)$, then the matrix $\lambda I - B$ is invertible. This proves that if λ is an eigenvalue of B, then $|\lambda| \leq \mathrm{s}(B)$. Note that equality is possible. For example, if $B = I$ then 1 is an eigenvalue of I. Also $\|I\| = \sqrt{N}$, so $\mathrm{s}(I) = \lim_{n\to\infty}\bigl(\sqrt{N}\bigr)^{1/n} = 1$. For a general matrix B, it can be even shown that there is always an eigenvalue λ such that $|\lambda| = \mathrm{s}(B)$. Hence, $s(B)$ is the maximal modulus of the eigenvalues of B.

(d) Show that the above series may converge, no matter how large $\|B\|$ is.

Solution: Consider the matrix $B = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$. Then $\|B\| = |a|$ can be as large as we like, but since $B^2 = 0$ we have $\mathrm{s}(B) = 0$ for all $a \in \mathbb{K}$. Hence there is no connection between the magnitude of $\|B\|$ and the convergence of the series under consideration in general.

***6.** Try to construct an array x_{jk} $(j, k \in \mathbb{N})$, such that the row and the column series converge absolutely, but $\sum_{i=0}^{\infty} x_{\sigma(i)}$ diverges for some bijection $\sigma\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

Solution: Consider the array

$$\begin{array}{cccccccc} 2 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Then we have

$$\sum_{j=0}^{\infty}\Bigl|\sum_{k=0}^{\infty} x_{jk}\Bigr| = \sum_{k=0}^{\infty}\Bigl|\sum_{j=0}^{\infty} x_{jk}\Bigr| = 1 + 0 + 0 + \cdots = 1,$$

showing that the row and column series converge absolutely. Now we sum a different way over the array. We take three 2 along the diagonal, then two -1 in the off diagonal, then again three 2 and two -1 and so on:

$$2+2+2-1-1+2+2+2-1-1+2+2+2-1-1+2+2+2-1-1+\dots.$$

It is evident that the latter series diverges.