THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 7 (Week 8)

MATH2962: Real and Complex Analysis (Advanced) — Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Let $f\colon D \to \mathbb{K}^N$ be a function. Give a formal definition of the following notions. Include the relevant assumptions on the domain and co-domain of the function.

(a) $\lim_{x\to a} f(x) = \infty$; (b) $\lim_{x\to -\infty} f(x) = \infty$; (c) $\lim_{x\to a+} f(x) = b$.

2. Decide whether the following sets are open or closed. Determine interior, closure and boundary.

(a) $[1,\infty) \subseteq \mathbb{R}$;

(b) $\{(-1)^n + 1/n \colon n \in \mathbb{N} \setminus \{0\}\} \subseteq \mathbb{R}$;

(c) $\{z \in \mathbb{C}\colon |z-1| + |z+1| < 4\} \subseteq \mathbb{C}$;

(d) $\mathbb{Z} \subseteq \mathbb{R}$;

(e) $\{(x,y,z)\colon x^2+y^2 < 1,\ x+y+z = 1\}$;

(f) $\{n/(n+1)\colon n \in \mathbb{N}\} \cup \{1\} \subseteq \mathbb{R}$.

3. Let X be a set and $(U_i)_{i\in I}$ be a family of subsets of X. The set I is an arbitrary index set (finite, countable or even uncountable depending on the problem).

By definition, the union of the family $(U_i)_{i\in I}$ is

$$\bigcup_{i\in I} U_i = \{x \in X\colon \text{there exists } i \in I \text{ such that } x \in U_i\} \tag{1}$$

and its intersection

$$\bigcap_{i\in I} U_i = \{x \in X\colon x \in U_i \text{ for all } i \in I\}. \tag{2}$$

Moreover, the complement of $U \subseteq X$ is given by

$$U^c := X \setminus U := \{x \in X\colon x \notin U\}.$$

Prove de *Morgan's law* asserting that

$$\Big(\bigcup_{i\in I} U_i\Big)^c = \bigcap_{i\in I} U_i^c \quad \text{and} \quad \Big(\bigcap_{i\in I} U_i\Big)^c = \bigcup_{i\in I} U_i^c. \tag{3}$$

Note: The sets A and B are equal if we prove that $A \subseteq B$ and $B \subseteq A$. Moreover, to establish that $A \subseteq B$, it is enough to show that $x \in A$ implies that $x \in B$ for every $x \in A$.

Extra questions for further practice

4. If $A \subset \mathbb{R}$ is a closed set bounded from above (below), show that A has a maximum (minimum).

5. Let $X \subseteq \mathbb{K}^N$ be an arbitrary set. A set $A \subseteq X$ is said to be *relatively open in* X if for every $x \in A$ there exists $r > 0$ such that $B(x, r) \cap X \subseteq A$. Moreover, $A \subseteq X$ is called *relatively closed in* X if its complement $A^c \cap X$ is relatively open in X.

(a) Which of the following sets are relatively open or closed in $X = [0, 2)$?

(i) $A = [0, 1)$ (ii) $A = [1, 2)$ (iii) $A = [1/2, 1)$

(b) Prove that $A \subseteq X$ is relatively open in X if and only if there exists an open set O in $\mathbb{K}^N$ with $A = X \cap O$.

(c) Show that $\emptyset$ and X are relatively open in X. Prove that arbitrary unions and finite intersections of relatively open sets in X are relatively open in X.

6. (a) If $A \subseteq B$, show that $\overline{A} \subseteq \overline{B}$ and that $\operatorname{int}(A) \subseteq \operatorname{int}(B)$.

*(b) Show that $\overline{A} \cup \overline{B} = \overline{A \cup B}$ and that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ with possibly proper inclusion.

Challenge questions (optional)

***7.** (a) Suppose $K_n \subseteq \mathbb{R}^N$ are closed non-empty sets such that $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$ and

$$\operatorname{diam}(K_n) := \sup_{x,y \in K_n} \|x - y\| \to 0 \quad \text{as } n \to \infty.$$

Show that $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$. (This is called *Cantor's Intersection Theorem.*)

(b) Give an example of non-empty open bounded sets O_n with $O_{n+1} \subseteq O_n$ and $\operatorname{diam}(O_n) \to 0$ as $n \to \infty$ such that $\bigcap_{n \in \mathbb{N}} O_n = \emptyset$.

***8.** Denote by $GL_N(\mathbb{K})$ the set of invertible matrices in $\mathbb{K}^{N \times N}$. We proved in Tutorial 6, Question **5** that $I - B$ is invertible if $\|B\| < 1$. Use this to prove that $GL_N(\mathbb{K})$ is open in $\mathbb{K}^{N \times N}$.

(*Note:* The above is a way to see that if we perturb the coefficients of an invertible matrix slightly, it will stay invertible. The proof does not make use of determinants, and the ideas apply to more general situations.)