THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial 7 (Week 8)

MATH2962: Real and Complex Analysis (Advanced) — Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Let $f\colon D \to \mathbb{K}^N$ be a function. Give a formal definition of the following notions. Include the relevant assumptions on the domain and co-domain of the function.

(a) $\lim\limits_{x\to a} f(x) = \infty$;

Solution: Suppose that $f\colon D \to \mathbb{R}$ and $a \in \overline{D}$ with $D \subset \mathbb{K}^d$. Then $f(x) \to \infty$ as $x \to a$ if for every $M \in \mathbb{R}$, there exists $\delta > 0$ such that $f(x) > M$ for all $x \in D$ with $0 < \|x - a\| < \delta$.

(b) $\lim\limits_{x\to -\infty} f(x) = \infty$;

Solution: Suppose that $f\colon D \to \mathbb{R}$ with $D \subset \mathbb{R}$ and $D \cap (-\infty, -n) \neq \emptyset$ for all $n \in \mathbb{N}$. Then $f(x) \to \infty$ as $x \to -\infty$ if for every $M \in \mathbb{R}$ there exists $x_0 \in \mathbb{R}$ such that $f(x) > M$ for all $x \in D$ with $x < x_0$.

(c) $\lim\limits_{x\to a+} f(x) = b$.

Solution: Suppose that $f\colon D \to \mathbb{K}^N$ with $D \subset \mathbb{R}$ and $(a, a + r) \cap D \neq \emptyset$ for all $r > 0$. We write $\lim\limits_{x\to a+} f(x) = b$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(x) - b\| < \varepsilon$ for all $x \in D \cap (a, a + \delta)$.

2. Decide whether the following sets are open or closed. Determine interior, closure and boundary.

(a) $[1, \infty) \subseteq \mathbb{R}$;

Solution: The set $[1, \infty)$ is closed since its complement $(-\infty, 1)$ is open. We have $\operatorname{int}[1, \infty) = (1, \infty)$ and the closure of $[1, \infty)$ is the set itself. The boundary of $[1, \infty)$ is $\{1\}$, which is obtained by subtracting $\operatorname{int}[1, \infty)$ from the closure of $[1, \infty)$.

(b) $\{(-1)^n + 1/n \colon n \in \mathbb{N} \setminus \{0\}\} \subseteq \mathbb{R}$;

Solution: The set is neither closed nor open. The closure is the sequence together with all points of accumulation. To find them, we denote $x_n = (-1)^n + 1/n$ for $n \geq 1$. We have $x_{2n} \to 1$ as $n \to \infty$ and $x_{2n+1} \to -1$ as $n \to \infty$. Hence, the accumulation points of $\{x_n : n \in \mathbb{N} \setminus \{0\}\}$ are 1 and -1. Hence, the closure of the given set is $\{(-1)^n + 1/n \colon n \in \mathbb{N} \setminus \{0\}\} \cup \{1, -1\}$. The interior is empty and the boundary is the same as its closure.

(c) $\{z \in \mathbb{C} \colon |z - 1| + |z + 1| < 4\} \subseteq \mathbb{C}$;

Solution: This is the interior of an ellipse with foci at $x = \pm 1$ without the boundary. Hence the set is open, the closure is the ellipse including the line bounding int, and the boundary is the ellipse $|z - 1| + |z + 1| = 4$.

(d) $\mathbb{Z} \subseteq \mathbb{R}$;

Solution: We see that the complement of $\mathbb{Z}$ in $\mathbb{R}$ is $\mathbb{R} \setminus \mathbb{Z} = \bigcup_{k\in\mathbb{Z}} (k, k + 1)$, which is an open set (as the union of open sets). This shows that $\mathbb{Z}$ is closed. Moreover, we have $\operatorname{int}(\mathbb{Z}) = \emptyset$ and $\partial\mathbb{Z} = \mathbb{Z}$.

(e) $\{(x, y, z) \colon x^2 + y^2 < 1,\, x + y + z = 1\}$;

Solution: Geometrically, the set is the part of the plane $x + y + z = 1$ within the open cylinder $x^2 + y^2 < 1$. Hence the set is neither open nor closed. Since it is part of a plane, its interior is empty. Its closure and boundary is $\{(x, y, z) \in \mathbb{R}^3 \colon x^2+y^2 \leq 1,\, x+y+z = 1\}$

(f) $\{n/(n + 1) \colon n \in \mathbb{N}\} \cup \{1\} \subseteq \mathbb{R}$.

Solution: We know that $n/(n + 1) \to 1$ as $n \to \infty$, so the set is closed. The reason is that every convergent sequence in the set has its limit in the set. The interior is empty since *every* interval about a point in the set intersects the complement of the set. The boundary equals the set itself.

3. Let X be a set and $(U_i)_{i\in I}$ be a family of subsets of X. The set I is an arbitrary index set (finite, countable or even uncountable depending on the problem).

By definition, the union of the family $(U_i)_{i\in I}$ is

$$\bigcup_{i\in I} U_i = \{x \in X \colon \text{there exists } i \in I \text{ such that } x \in U_i\} \tag{1}$$

and its intersection

$$\bigcap_{i\in I} U_i = \{x \in X \colon x \in U_i \text{ for all } i \in I\}. \tag{2}$$

Moreover, the complement of $U \subseteq X$ is given by

$$U^c := X \setminus U := \{x \in X \colon x \notin U\}.$$

Prove de *Morgan's law* asserting that

$$\Big(\bigcup_{i\in I} U_i\Big)^c = \bigcap_{i\in I} U_i^c \quad\text{and}\quad \Big(\bigcap_{i\in I} U_i\Big)^c = \bigcup_{i\in I} U_i^c. \tag{3}$$

Note: The sets A and B are equal if we prove that $A \subseteq B$ and $B \subseteq A$. Moreover, to establish that $A \subseteq B$, it is enough to show that $x \in A$ implies that $x \in B$ for every $x \in A$.

Solution: We show the first identity in (3). Let $x \in \big(\bigcup_{i\in I} U_i\big)^c$ be arbitrary. Then $x \notin \bigcup_{i\in I} U_i$. From (1), we have that $x \notin U_i$ for all $i \in I$. Thus, $x \in U_i^c$ for all $i \in I$, which means that $x \in \bigcap_{i\in I} U_i^c$. This proves that $\big(\bigcup_{i\in I} U_i\big)^c \subseteq \bigcap_{i\in I} U_i^c$. We now show the converse inclusion. Let $x \in \bigcap_{i\in I} U_i^c$. By (2), we have $x \in U_i^c$ for all $i \in I$ and so $x \notin U_i$ for all $i \in I$. By (1), we conclude that $x \notin \bigcup_{i\in I} U_i$, that is, $x \in (\bigcup_{i\in I} U_i)^c$. Hence $\bigcap_{i\in I} U_i^c \subseteq \big(\bigcup_{i\in I} U_i\big)^c$.

For the second identity in (3), we can use the first identity (with U_i^c instead of U_i) and the fact that $(A^c)^c = A$ for every $A \subseteq X$. Indeed, we have

$$\Big(\bigcup_{i\in I} U_i^c\Big)^c = \bigcap_{i\in I} (U_i^c)^c = \bigcap_{i\in I} U_i.$$

By taking the complement in the above equality, we get

$$\bigcup_{i\in I} U_i^c = \Big(\bigcap_{i\in I} U_i\Big)^c.$$

Extra questions for further practice

4. If $A \subset \mathbb{R}$ is a closed set bounded from above (below), show that A has a maximum (minimum).

Solution: Suppose that $A \subset \mathbb{R}$ is bounded from above. Then, by the least upper bound axiom, $M := \sup A$ exists (in $\mathbb{R}$). By the definition of a supremum, we infer that for every $n \in \mathbb{N} \setminus \{0\}$, there exists $a_n \in A$ such that

$$M - \frac{1}{n} < a_n \leq M.$$

Since $1/n \to 0$ as $n \to \infty$, by the squeeze law, we conclude that $a_n \to M$ as $n \to \infty$. Since A is a closed set, it must contain all its accumulation points. In particular, $M \in A$. Hence, M is the maximum of A.

5. Let $X \subseteq \mathbb{K}^N$ be an arbitrary set. A set $A \subseteq X$ is said to be *relatively open in* X if for every $x \in A$ there exists $r > 0$ such that $B(x, r) \cap X \subseteq A$. Moreover, $A \subseteq X$ is called *relatively closed in* X if its complement $A^c \cap X$ is relatively open in X.

(a) Which of the following sets are relatively open or closed in $X = [0, 2)$?

(i) $A = [0, 1)$

Solution: The set is relatively open in $[0, 2)$, because about every point x in $[0, 1)$ there is an open interval whose intersection with $[0, 2)$ is *entirely* in $[0, 1)$. Note that $A^c \cap X = [1, 2)$ is not relatively open in X since $1 \in A^c \cap X$ and there is no small interval about 1 entirely contained in $A^c \cap X$. Hence, $[0, 1)$ is not relatively closed in X.

(ii) $A = [1, 2)$

Solution: The set $[1, 2)$ is relatively closed in $X = [0, 2)$ since its complement in X, namely $[0, 1)$, is relatively open in X. However, $[1, 2)$ is not relatively open in $[0, 2)$ since $[0, 1)$ is not relatively closed in X.

(iii) $A = [1/2, 1)$

Solution: The set $[1/2, 1)$ is not relatively open in $[0, 2)$ because there is no small interval about $1/2$ entirely contained in the intersection $A \cap X$. The set $A = [1/2, 1)$ is also not relatively closed in $[0, 2)$ because $A^c \cap X = [0, 1/2) \cup [1, 2)$ is not relatively open in X. Indeed, there is no small interval about 1 entirely contained in the intersection $A^c \cap X$.

(b) Prove that $A \subseteq X$ is relatively open in X if and only if there exists an open set O in $\mathbb{K}^N$ with $A = X \cap O$.

Solution: "$\Longrightarrow$" We first prove that if A is relatively open in X, then there exists an open set O in $\mathbb{K}^N$ such that $A = X \cap O$. Since A is relatively open in X, we have that for every $x \in A$, there exists $r_x > 0$ such that $B(x, r_x) \cap X \subseteq A$. Hence, we have $\bigcup_{x \in A} (B(x, r_x) \cap X) \subseteq A$, that is $\left(\bigcup_{x \in A} B(x, r_x)\right) \cap X \subseteq A$. Since an arbitrary union of open sets is open, we obtain that

$$O := \bigcup_{x \in A} B(x, r_x) \tag{4}$$

is an open set. This proves that $O \cap X \subseteq A$. From (4) and $A \subseteq X$, we also have $A \subseteq O \cap X$. Hence, we conclude that $A = X \cap O$.

"$\Longleftarrow$" We now show that if O is an open set with $A = X \cap O$, then A is relatively open in X. Indeed, for every $x \in A$, we have $x \in O$. Using that O is open, we infer that there exists $r > 0$ such that $B(x, r) \subseteq O$. Hence $B(x, r) \cap X \subseteq O \cap X = A$, proving that A is relatively open in X.

(c) Show that $\emptyset$ and X are relatively open in X. Prove that arbitrary unions and finite intersections of relatively open sets in X are relatively open in X.

Solution: We have $X = X \cap \mathbb{K}^N$, where $\mathbb{K}^N$ is open. Therefore, using the characterisation of relatively open sets from part (b), we conclude that X is relatively open in X. Similarly, $\emptyset$ is relatively open in X since $\emptyset \cap X = \emptyset$ and $\emptyset$ is open in $\mathbb{K}^N$.

Let $(O_\alpha)_{\alpha \in I}$ be an arbitrary family of relatively open sets in X, where I is any index set. We prove that

$$\bigcup_{\alpha \in I} O_\alpha \quad \text{is relatively open in } X. \tag{5}$$

Let $\alpha \in I$ be arbitrary. Since O_α is relatively open in X, then by part (b), there exists an open set $\widetilde{O}_\alpha \subseteq \mathbb{K}^N$ with $\widetilde{O}_\alpha \cap X = O_\alpha$. We find that

$$\bigcup_{\alpha \in I} O_\alpha = \left(\bigcup_{\alpha \in I} \widetilde{O}_\alpha \right) \cap X. \tag{6}$$

Since $\bigcup_{\alpha \in I} \widetilde{O}_\alpha$ is open in $\mathbb{K}^N$ (as an arbitrary union of open sets in $\mathbb{K}^N$), from (6) and part (b) we conclude the assertion of (5). In a similar way, we can prove that any *finite intersection* of relatively open sets in X is relatively open in X.

6. (a) If $A \subseteq B$, show that $\overline{A} \subseteq \overline{B}$ and that $\operatorname{int}(A) \subseteq \operatorname{int}(B)$.

Solution: Suppose that $x \in \overline{A}$. Then $B(x,r) \cap A \neq \emptyset$ for all $r > 0$. Since $A \subseteq B$, we also have $B(x,r) \cap B \neq \emptyset$ for all $r > 0$. Hence $x \in \overline{B}$, proving that $\overline{A} \subseteq \overline{B}$. Suppose now that $x \in \operatorname{int}(A)$. Then, there exists $r > 0$ such that $B(x,r) \subseteq A$, and thus $B(x,r) \subseteq B$ since $A \subseteq B$. This means that $x \in \operatorname{int}(B)$, showing that $\operatorname{int}(A) \subseteq \operatorname{int}(B)$.

*(b) Show that $\overline{A} \cup \overline{B} = \overline{A \cup B}$ and that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ with possibly proper inclusion.

Solution: We first prove that $\overline{A} \cup \overline{B} = \overline{A \cup B}$, that is

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B} \quad \text{and} \quad \overline{A \cup B} \subseteq \overline{A} \cup \overline{B}. \tag{7}$$

To prove the first inclusion in (7), we fix $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$. Hence $B(x,r) \cap A \neq \emptyset$ or $B(x,r) \cap B \neq \emptyset$ for all $r > 0$, that is, $B(x,r) \cap (A \cup B) \neq \emptyset$ for all $r > 0$. Hence $x \in \overline{A \cup B}$, showing that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. For the second inclusion, we see that $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ so that $A \cup B \subseteq \overline{A} \cup \overline{B}$. As a finite union of closed sets is closed, we infer that $\overline{A} \cup \overline{B}$ is closed. Therefore, by (a) we have

$$\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B},$$

which proves the second inclusion in (7).

We now prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and give an example for which this inclusion is proper. We note that $A \cap B \subseteq \overline{A} \cap \overline{B}$. As an intersection of closed sets is closed, we have that $\overline{A} \cap \overline{B}$ is closed. Thus by (a), we obtain that

$$\overline{A \cap B} \subseteq \overline{\overline{A} \cap \overline{B}} = \overline{A} \cap \overline{B}.$$

To see that there is no equality in general, we consider $A = (0,1)$ and $B := (1,2)$. Then $A \cap B = \emptyset$, implying that $\overline{A \cap B} = \emptyset$. On the other hand, $\overline{A} = [0,1]$ and $\overline{B} := [1,2]$, so that $\overline{A} \cap \overline{B} = \{1\}$. In this case, $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ because $\{1\} \not\subseteq \emptyset$.

Challenge questions (optional)

***7.** (a) Suppose $K_n \subseteq \mathbb{R}^N$ are closed non-empty sets such that $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$ and

$$\operatorname{diam}(K_n) := \sup_{x,y \in K_n} \|x - y\| \to 0 \quad \text{as } n \to \infty.$$

Show that $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$. (This is called *Cantor's Intersection Theorem.*)

Solution: For every $n \in \mathbb{N}$, we can choose $x_n \in K_n$ since K_n is non-empty. The proof of $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$ is divided into two steps.

Step 1: *We show that (x_n) is Cauchy sequence.*
To this aim, we prove that for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|x_m - x_n\| < \varepsilon \quad \text{for every } m > n > n_\varepsilon. \tag{8}$$

By assumption, we have $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$, implying that $x_m \in K_m \subseteq K_n$ for all $m > n$. Since $x_m, x_n \in K_n$, we have

$$\|x_m - x_n\| \leq \sup_{x,y \in K_n} \|x - y\| = \operatorname{diam}(K_n) \quad \text{for all } m > n. \tag{9}$$

From $\operatorname{diam}(K_n) \to 0$ as $n \to \infty$, we infer that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\operatorname{diam}(K_n) < \varepsilon$ for all $n > n_\varepsilon$. This, jointly with (9), proves the claim of (8). Hence, (x_n) is a Cauchy sequence. Thus, by the Completeness Axiom of $\mathbb{R}^N$, the sequence (x_n) converges to some x in $\mathbb{R}^N$.

Step 2: *We show that* $x \in \bigcap_{n\in\mathbb{N}} K_n$.
We know from above that $x_m \in K_n$ for all $m > n$. As K_n is closed and $(x_m)_{m>n}$ is a sequence in K_n converging to x, we must have $x \in K_n$. Since this is true for all $n \in \mathbb{N}$, we conclude that $x \in \bigcap_{n\in\mathbb{N}} K_n$, which completes the proof.

(b) Give an example of non-empty open bounded sets O_n with $O_{n+1} \subseteq O_n$ and $\operatorname{diam}(O_n) \to 0$ as $n \to \infty$ such that $\bigcap_{n\in\mathbb{N}} O_n = \emptyset$.

Solution: For every $n \in \mathbb{N}$, we define $O_n = (0, \frac{1}{n+1}) \subseteq \mathbb{R}$. Then clearly O_n is open, $O_{n+1} \subseteq O_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} O_n = \emptyset$.

***8.** Denote by $GL_N(\mathbb{K})$ the set of invertible matrices in $\mathbb{K}^{N\times N}$. We proved in Tutorial 6, Question **5** that $I - B$ is invertible if $\|B\| < 1$. Use this to prove that $GL_N(\mathbb{K})$ is open in $\mathbb{K}^{N\times N}$.

(*Note:* The above is a way to see that if we perturb the coefficients of an invertible matrix slightly, it will stay invertible. The proof does not make use of determinants, and the ideas apply to more general situations.)

Solution: Let A be any invertible matrix in $\mathbb{K}^{N\times N}$. We show below that there exists $r > 0$ such that the open ball about A with radius r is contained in $GL_N(\mathbb{K})$. In other words, there exists $r > 0$ such that every matrix B in $\mathbb{K}^{N\times N}$ is invertible when $\|A - B\| < r$. Note that B in $\mathbb{K}^{N\times N}$ is invertible if and only if $A^{-1}B$ is invertible. We write

$$A^{-1}B = I - (I - A^{-1}B)$$

so that $A^{-1}B$ is invertible if

$$\|I - A^{-1}B\| < 1. \tag{10}$$

Using that

$$\|I - A^{-1}B\| = \|A^{-1}(A - B)\| \leq \|A^{-1}\|\,\|A - B\|,$$

we conclude that if

$$\|A^{-1}\|\,\|A - B\| < 1,$$

then (10) holds so that $A^{-1}B$ is invertible implying that B is invertible. Thus if $r = 1/\|A^{-1}\|$, then every matrix B in $\mathbb{K}^{N\times N}$ is invertible when $\|A - B\| < r$. So, $GL_N(\mathbb{K})$ is open in $\mathbb{K}^{N\times N}$.