

Tutorial 8 (Week 9)

MATH2962: Real and Complex Analysis (Advanced)

Semester 1, 2012

Web Page: <http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/>

Lecturer: Florica Cirstea

Questions marked with \* are more difficult questions.

Questions to complete during the tutorial

1. Suppose that  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  converges for all  $z \in \mathbb{C}$  with  $|z| < \rho$ . Show that  $\lim_{z \rightarrow 0} f(z) = a_0$ .

2. Compute the following limits. Always assume that  $\alpha > 0$  and work with the series expansion

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Observe that  $\exp(x) \geq x^k/k!$  for all  $k \geq 0$  and all  $x \geq 0$ .

- (a)  $\lim_{x \rightarrow \infty} \exp(x) = \infty$ ;      (c)  $\lim_{x \rightarrow \infty} x^\alpha a^x$ ,  $a \in (0, 1)$ ;      (e)  $\lim_{x \rightarrow 0^+} x^\alpha \log x$ ;  
(b)  $\lim_{x \rightarrow \infty} x^{-\alpha}$ ;      (d)  $\lim_{x \rightarrow \infty} x^{-\alpha} \log x$ ;      (f)  $\lim_{x \rightarrow \infty} x^{1/x}$ ;

3. Suppose that  $\alpha \in \mathbb{R}$ . For  $k \in \mathbb{N}$ , set

$$\binom{\alpha}{0} := 1 \quad \text{and} \quad \binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!} \quad \text{if } k \geq 1.$$

Consider the *binomial series*  $f(x) := \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ .

- (a) Show that  $\binom{\alpha-1}{k} + \binom{\alpha-1}{k-1} = \binom{\alpha}{k}$  and  $k \binom{\alpha}{k} = \alpha \binom{\alpha-1}{k-1}$  for all  $k \geq 1$ .  
(b) Prove that the power series  $f(x)$  converges for  $|x| < 1$ . What about if  $\alpha \in \mathbb{N}$ ?  
(c) Show that  $(1+x)f'(x) = \alpha f(x)$  whenever  $|x| < 1$ .  
(d) Set  $g(x) := (1+x)^\alpha$  and show that  $(1+x)g'(x) = \alpha g(x)$ .  
(e) Use (c) and (d) to show that  $f(x)/g(x) = 1$  for all  $|x| < 1$  and deduce that for  $|x| < 1$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

Extra questions for further practice

4. Suppose  $g: \mathbb{R} \rightarrow \mathbb{R}$  is monotone. Show that  $\lim_{t \rightarrow s^+} g(t)$  and  $\lim_{t \rightarrow s^-} g(t)$  exist for every  $s \in \mathbb{R}$ .

5. Suppose that  $D \subseteq \mathbb{K}^N$  and  $f: D \rightarrow \mathbb{R}$  is a function. Fix  $x_0 \in \overline{D}$ .

- (a) Explain why  $g(r) := \inf_{x \in B(x_0, r) \cap D} f(x)$  and  $h(r) := \sup_{x \in B(x_0, r) \cap D} f(x)$  are monotone functions of  $r > 0$ .

(b) By Question 4, the limits  $\lim_{r \rightarrow 0^+} g(r)$  and  $\lim_{r \rightarrow 0^+} h(r)$  exist. We define

$$\begin{aligned}\liminf_{x \rightarrow x_0} f(x) &:= \lim_{r \rightarrow 0^+} g(r) = \lim_{r \rightarrow 0^+} \left( \inf_{x \in B(x_0, r) \cap D} f(x) \right) \\ \limsup_{x \rightarrow x_0} f(x) &:= \lim_{r \rightarrow 0^+} h(r) = \lim_{r \rightarrow 0^+} \left( \sup_{x \in B(x_0, r) \cap D} f(x) \right).\end{aligned}$$

To answer the questions below, adapt the proofs of the corresponding facts for sequences from the lecture notes.

- (i) If  $\liminf_{x \rightarrow x_0} f(x) = \limsup_{x \rightarrow x_0} f(x) = b \in \mathbb{R}$ , show that  $\lim_{x \rightarrow x_0} f(x) = b$  exists.
- (ii) If  $\lim_{x \rightarrow x_0} f(x) = b$  exists, show that  $\liminf_{x \rightarrow x_0} f(x) = \limsup_{x \rightarrow x_0} f(x) = b$ .

6. Define  $\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$ . We know that  $\exp(x+y) = \exp(x)\exp(y)$  for all  $x, y \in \mathbb{C}$ .

- (a) Show that  $\exp$  is continuous on  $\mathbb{C}$  and that  $\exp(0) = 1$  and  $\exp(1) = e$ .
- (b) Show that  $\exp(z) \neq 0$  for all  $z \in \mathbb{C}$  and  $\exp(-z) = 1/\exp(z)$ .
- (c) Show that  $\exp(nz) = (\exp(z))^n$  for all  $n \in \mathbb{Z}$  and  $\exp(z/n) = \sqrt[n]{\exp(z)}$  for all  $n \in \mathbb{N} \setminus \{0\}$ . In particular,  $\exp(n) = e^n$  for all  $n \in \mathbb{Z}$  and  $\exp(1/n) = \sqrt[n]{e}$  for all  $n \in \mathbb{N} \setminus \{0\}$ .
- (d) Show that  $\overline{\exp(z)} = \exp(\bar{z})$  for all  $z \in \mathbb{C}$ .
- (e) Show that  $|\exp(it)| = 1$  for all  $t \in \mathbb{R}$ .

### Challenge questions (optional)

7. Let  $D \subseteq \mathbb{K}^N$  and  $f, g: D \rightarrow \mathbb{R}$  and  $x_0 \in \bar{D}$ . If  $D = \mathbb{R}$  or a suitable unbounded interval we also admit  $x_0 = \pm\infty$ . We introduce *big Oh* and *little Oh* notation to compare the asymptotic behaviour of functions as  $x \rightarrow x_0$ . We say

$$\begin{cases} f(x) = O(g(x)) \text{ as } x \rightarrow x_0 & \text{if } \limsup_{x \rightarrow x_0} \frac{|f(x)|}{|g(x)|} < \infty, \\ f(x) = o(g(x)) \text{ as } x \rightarrow x_0 & \text{if } \lim_{x \rightarrow x_0} \frac{|f(x)|}{|g(x)|} = 0. \end{cases}$$

- (a) Let  $f_1(x) = O(g(x))$  and  $f_2(x) = O(g(x))$  as  $x \rightarrow x_0$ . Show that  $\alpha f_1(x) + \beta f_2(x) = O(g(x))$  as  $x \rightarrow x_0$  for all  $\alpha, \beta \in \mathbb{R}$ . Prove a similar statement if  $O$  is replaced by  $o$ .
- (b) Let  $f_1(x) = O(g_1(x))$  and  $f_2(x) = O(g_2(x))$  as  $x \rightarrow x_0$ . Show that  $f_1(x)f_2(x) = O(g_1(x)g_2(x))$  as  $x \rightarrow x_0$ . Prove a similar statement if  $O$  is replaced by  $o$ .
- (c) Let  $f_1(x) = O(g_1(x))$  and  $f_2(x) = o(g_2(x))$  as  $x \rightarrow x_0$ . Show that  $f_1(x)f_2(x) = o(g_1(x)g_2(x))$  as  $x \rightarrow x_0$ .
- (d) Let  $f(x) = O(g(x))$  as  $x \rightarrow x_0$  and  $\alpha > 0$ . Show that  $|f(x)|^\alpha = O(|g(x)|^\alpha)$  as  $x \rightarrow x_0$ .