THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial 8 (Week 9)

MATH2962: Real and Complex Analysis (Advanced) Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ converges for all $z \in \mathbb{C}$ with $|z| < \varrho$. Show that $\lim_{z\to 0} f(z) = a_0$.

Solution: We want to prove that $\|f(z) - a_0\|$ converges to 0 as $z \in \mathbb{C}$ approaches 0. Let $r \in (0, \varrho)$ be fixed. For every $z \in \mathbb{C}$ with $0 < |z| \le r$, we have

$$\|f(z) - a_0\| = \Big\|\sum_{k=1}^{\infty} a_k z^k\Big\| \le |z| \sum_{k=1}^{\infty} \|a_k\| |z|^{k-1} \le |z| \sum_{k=1}^{\infty} \|a_k\| r^{k-1} \le \frac{|z|}{r} \sum_{k=0}^{\infty} \|a_k\| r^k.$$

Since the power series $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely for any $z \in \mathbb{C}$ with $|z| < \varrho$, we have that the power series $\sum_{k=0}^{\infty} \|a_k\| r^k$ converges (since $0 < r < \varrho$). Hence, if we let $z \to 0$, we find that

$$0 \le \|f(z) - a_0\| \le \frac{|z|}{r} \sum_{k=0}^{\infty} \|a_k\| r^k \to 0,$$

The claim follows from the squeeze law.

2. Compute the following limits. Always assume that $\alpha > 0$ and work with the series expansion

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Observe that $\exp(x) \ge x^k/k!$ for all $k \ge 0$ and all $x \ge 0$.

(a) $\lim_{x\to\infty} \exp(x) = \infty$;

Solution: This is obvious since $\exp(x) \ge x \to \infty$ as $x \to \infty$.

(b) $\lim_{x\to\infty} x^{-\alpha}$;

Solution: By the definition of general powers, we have $x^{-\alpha} = \dfrac{1}{\exp(\alpha \log x)}$. Since $\log x \ge 0$ for $x \ge 1$, we have $\exp(\alpha \log x) \ge \alpha \log x$ and thus $x^{-\alpha} \to 0$ as $x \to \infty$.

(c) $\lim_{x\to\infty} x^{\alpha} a^x$, $a \in (0, 1)$;

Solution: We can rewrite the function as $x^{\alpha} a^x = \dfrac{x^{\alpha}}{\exp(-x \log a)}$. Since $a \in (0, 1)$, we have $\log a < 0$. It follows that

$$\exp(-x \log a) = \exp(x|\log a|) \ge \frac{x^k |\log a|^k}{k!}$$

for all $k \geq 0$. If we choose $k > \alpha$, then

$$0 < x^\alpha a^x = \frac{x^\alpha}{\exp(-x\log a)} \leq \frac{x^\alpha}{\frac{1}{k!}x^k|\log a|^k} = \frac{k!}{|\log a|^k}x^{\alpha-k} \to 0$$

as $x \to \infty$ by (b). Hence, by the squeeze law, we find that $x^\alpha a^x \to 0$ as $x \to \infty$.

(d) $\lim\limits_{x\to\infty} x^{-\alpha}\log x$;

Solution: Since

$$x^\alpha = \exp(\alpha\log x) \geq \frac{1}{2}(\alpha\log x)^2$$

for all $x > 1$ we get

$$0 < x^{-\alpha}\log x = \frac{\log x}{x^\alpha} \leq \frac{2\log x}{(\alpha\log x)^2} = \frac{2}{\alpha^2\log x}$$

for all $x > 1$. Since $\log x \to \infty$ as $x \to \infty$, we conclude that $x^{-\alpha}\log x \to 0$ as $x \to \infty$.

(e) $\lim\limits_{x\to 0+} x^\alpha\log x$;

Solution: We have $x^\alpha\log x = \dfrac{-\log(1/x)}{(1/x)^\alpha}$ for all $x > 0$. Since $1/x \to \infty$ as $x \to 0+$ we get from part (d) that the limit is zero as $x \to 0+$.

(f) $\lim\limits_{x\to\infty} x^{1/x}$;

Solution: By the continuity of the exponential function at zero and part (d) we have $x^{1/x} = \exp\Big(\dfrac{\log x}{x}\Big) \to \exp(0) = 1$ as $x \to \infty$.

3. Suppose that $\alpha \in \mathbb{R}$. For $k \in \mathbb{N}$, set

$$\binom{\alpha}{0} := 1 \quad\text{and}\quad \binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!} \quad \text{if } k \geq 1.$$

Consider the *binomial series* $f(x) := \sum\limits_{k=0}^{\infty}\dbinom{\alpha}{k}x^k$.

(a) Show that $\dbinom{\alpha-1}{k} + \dbinom{\alpha-1}{k-1} = \dbinom{\alpha}{k}$ and $k\dbinom{\alpha}{k} = \alpha\dbinom{\alpha-1}{k-1}$ for all $k \geq 1$.

Solution: If $k = 1$, then the claim of (a) is obvious since $\dbinom{\alpha}{1} = \alpha$ and $\dbinom{\alpha-1}{0} = 1$. We thus assume that $k \geq 2$. By the definition of $\dbinom{\alpha}{k}$, we have

$$\begin{aligned}\binom{\alpha}{k} - \binom{\alpha-1}{k} &= \frac{1}{k!}\Big(\alpha(\alpha-1)\dots(\alpha-k+1) - (\alpha-1)(\alpha-2)\dots(\alpha-k)\Big)\\ &= \frac{1}{k!}(\alpha-1)(\alpha-2)\dots(\alpha-k+1)\Big(\alpha-(\alpha-k)\Big)\\ &= \frac{1}{(k-1)!}(\alpha-1)(\alpha-2)\dots\big((\alpha-1)-(k-1)+1\big) = \binom{\alpha-1}{k-1}.\end{aligned}$$

Similarly, we find that

$$\begin{aligned}k\binom{\alpha}{k} &= \frac{k}{k!}\alpha(\alpha-1)\dots(\alpha-k+1)\\ &= \alpha\frac{1}{(k-1)!}(\alpha-1)(\alpha-2)\dots(\alpha-1-(k-1)+1) = \alpha\binom{\alpha-1}{k-1}\end{aligned}$$

for all $k \geq 2$. It does *not* make sense to write $\dbinom{\alpha}{k} = \dfrac{\alpha!}{k!(\alpha-k)!}$ if $\alpha \notin \mathbb{N}$!

(b) Prove that the power series $f(x)$ converges for $|x|<1$. What about if $\alpha \in \mathbb{N}$?

Solution: If $\alpha \in \mathbb{N}$, then $\binom{\alpha}{k} = 0$ for $k > \alpha$ and $\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ reduces to the binomial formula valid for all $x \in \mathbb{R}$. Now assume that $\alpha \notin \mathbb{N}$. Then $\binom{\alpha}{k} \neq 0$ for all $k \in \mathbb{N}$. Moreover, we have

$$\frac{\left|\binom{\alpha}{k+1}\right|}{\left|\binom{\alpha}{k}\right|} = \frac{|\alpha - k|}{k+1} \to 1 \quad \text{as } k \to \infty.$$

Hence, by the ratio test (see Tutorial 6, Question **1**), the radius of convergence of the binomial series is $\varrho = 1/1 = 1$.

(c) Show that $(1+x)f'(x) = \alpha f(x)$ whenever $|x|<1$.

Solution: We know that we can differentiate power series term-by-term. Hence

$$f'(x) = \sum_{k=1}^{\infty} k \binom{\alpha}{k} x^{k-1}.$$

By using (a), we obtain that

$$\begin{aligned}(1+x)f'(x) &= \alpha \sum_{k=1}^{\infty} \binom{\alpha-1}{k-1} x^{k-1} + \alpha \sum_{k=1}^{\infty} \binom{\alpha-1}{k-1} x^k \\ &= \alpha \sum_{k=0}^{\infty} \binom{\alpha-1}{k} x^k + \alpha \sum_{k=1}^{\infty} \binom{\alpha-1}{k-1} x^k \\ &= \alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k - \alpha \sum_{k=1}^{\infty} \binom{\alpha-1}{k-1} x^k + \alpha \sum_{k=1}^{\infty} \binom{\alpha-1}{k-1} x^k \\ &= \alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = \alpha f(x).\end{aligned}$$

(d) Set $g(x) := (1+x)^\alpha$ and show that $(1+x)g'(x) = \alpha g(x)$.

Solution: We clearly have

$$(1+x)g'(x) = \alpha(1+x)(1+x)^{\alpha-1} = \alpha(1+x)^\alpha = \alpha g(x).$$

(e) Use (c) and (d) to show that $f(x)/g(x) = 1$ for all $|x|<1$ and deduce that for $|x|<1$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

Solution: By the quotient rule, we have

$$\left(\frac{f}{g}\right)' = \frac{1}{g^2}(f'g - fg').$$

By part (c) and (d), we have $(1+x)f'g = (1+x)fg' = \alpha f(x)g(x)$ for $|x|<1$. This implies that $fg' = f'g$. Hence

$$\left(\frac{f}{g}\right)' = 0$$

on the set $|x|<1$, which implies that $f(x)/g(x) = c$ is constant. For $x = 0$ we have $f(0)/g(0) = 1 = c$. Consequently, $f(x) = g(x)$ for $|x|<1$ as claimed.

Extra questions for further practice

4. Suppose $g\colon \mathbb{R} \to \mathbb{R}$ is monotone. Show that $\lim_{t\to s+} g(t)$ and $\lim_{t\to s-} g(t)$ exist for every $s \in \mathbb{R}$.

Solution: We give a proof similar to that for the corresponding result for sequences (bounded monotone sequences converge). Assume that g is increasing and fix $s \in \mathbb{R}$. Then $\{g(t)\colon t < s\}$ is bounded from above by $g(s)$. Hence

$$b := \sup\{g(t)\colon t < s\} < \infty$$

exists. Fix $\varepsilon > 0$. Then by definition of a supremum, there exists $t_0 < s$ so that $b - \varepsilon < g(t_0)$. Since b is the supremum of $\{g(t)\colon t < s\}$ and g is increasing, we conclude that

$$b - \varepsilon < g(t_0) \leq g(t) \leq b < b + \varepsilon$$

for all $t \in (t_0, s)$ and therefore $|g(t) - b| < \varepsilon$ whenever $0 < s - t < \delta := s - t_0$. Since the above argument works for every choice of $\varepsilon > 0$, we obtain that $g(t) \to b$ as $t \to s-$. Use similar arguments for the other cases.

5. Suppose that $D \subseteq \mathbb{K}^N$ and $f\colon D \to \mathbb{R}$ is a function. Fix $x_0 \in \overline{D}$.

(a) Explain why $g(r) := \inf_{x\in B(x_0,r)\cap D} f(x)$ and $h(r) := \sup_{x\in B(x_0,r)\cap D} f(x)$ are monotone functions of $r > 0$.

Solution: Set $A_r := \{f(x)\colon x \in B(x_0, r) \cap D\}$. Then clearly $A_r \subseteq A_s$ for $0 < r \leq s$. Hence by elementary properties of supremum and infimum

$$g(r) = \inf A_r \geq \inf A_s = g(s) \qquad \text{and} \qquad h(r) = \sup A_r \leq \sup A_s = h(s)$$

if $0 < r \leq s$.

(b) By Question 4, the limits $\lim_{r\to 0+} g(r)$ and $\lim_{r\to 0+} h(r)$ exist. We define

$$\begin{aligned}\liminf_{x\to x_0} f(x) &:= \lim_{r\to 0+} g(r) = \lim_{r\to 0+}\Bigl(\inf_{x\in B(x_0,r)\cap D} f(x)\Bigr)\\ \limsup_{x\to x_0} f(x) &:= \lim_{r\to 0+} h(r) = \lim_{r\to 0+}\Bigl(\sup_{x\in B(x_0,r)\cap D} f(x)\Bigr).\end{aligned}$$

To answer the questions below, adapt the proofs of the corresponding facts for sequences from the lecture notes.

(i) If $\liminf_{x\to x_0} f(x) = \limsup_{x\to x_0} f(x) = b \in \mathbb{R}$, show that $\lim_{x\to x_0} f(x) = b$ exists.

Solution: By definition infimum and supremum

$$g(r) := \inf_{x\in B(x_0,r)\cap D} f(x) \leq f(y) \leq \sup_{x\in B(x_0,r)\cap D} f(x) =: h(r)$$

for all $y \in B(x_0, r) \cap D$. Fix $\varepsilon > 0$. By the assumption and the definition of limit inferior and limit superior there exists $\delta > 0$ so that $|g(r) - b| < \varepsilon$ and $|h(r) - b| < \varepsilon$ for all $0 < r < \delta$. Hence from the above

$$-\varepsilon < g(r) - b < f(y) - b < h(r) - b < \varepsilon$$

whenever $y \in D$ with $|y - x_0| < \delta$. Hence $f(x) \to b$ as $x \to x_0$, since the above argument works for every choice of $\varepsilon > 0$.

(ii) If $\lim_{x\to x_0} f(x) = b$ exists, show that $\liminf_{x\to x_0} f(x) = \limsup_{x\to x_0} f(x) = b$.

Solution: Fix $\varepsilon > 0$. By definition of a limit, there exists $\delta > 0$ such that

$$|f(x) - b| < \varepsilon/2$$

or equivalently

$$b - \varepsilon/2 < f(x) < b + \varepsilon/2$$

for all $x \in B(x_0, \delta) \cap D$. Hence

$$-\varepsilon/2 + b \leq g(r) := \inf_{x \in B(x_0,r)\cap D} f(x) \leq \sup_{x \in B(x_0,r)\cap D} f(x) =: h(r) \leq b + \varepsilon/2$$

for all $r \in (0, \delta)$. Therefore, we obtain that

$$|g(r) - b| < \varepsilon \qquad \text{and} \qquad |h(r) - b| < \varepsilon$$

for all $r \in (0, \delta)$. Since the above arguments work for every choice of $\varepsilon > 0$, we conclude that $g(r) \to b$ and $h(r) \to b$ as $r \to 0+$.

6. Define $\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$. We know that $\exp(x + y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{C}$.

(a) Show that exp is continuous on $\mathbb{C}$ and that $\exp(0) = 1$ and $\exp(1) = e$.

Solution: Clearly $\exp(0) = 1$ and $\exp(1) := \sum_{k=0}^{\infty} \frac{1}{k!} = e$ as shown on Tutorial 2 Question **2**. For the continuity, note that

$$\exp(z + h) - \exp(z) = \exp(z)\bigl(\exp(h) - 1\bigr) \to 0$$

as $h \to 0$ by Question 1, proving that exp is continuous.

(b) Show that $\exp(z) \neq 0$ for all $z \in \mathbb{C}$ and $\exp(-z) = 1/\exp(z)$.

Solution: From the above $1 = \exp(0) = \exp(z - z) = \exp(z)\exp(-z)$ for all $z \in \mathbb{C}$. Hence $\exp(z) \neq 0$ and $\exp(-z) = 1/\exp(z)$.

(c) Show that $\exp(nz) = (\exp(z))^n$ for all $n \in \mathbb{Z}$ and $\exp(z/n) = \sqrt[n]{\exp(z)}$ for all $n \in \mathbb{N} \setminus \{0\}$. In particular, $\exp(n) = e^n$ for all $n \in \mathbb{Z}$ and $\exp(1/n) = \sqrt[n]{e}$ for all $n \in \mathbb{N} \setminus \{0\}$.

Solution: If $n \in \mathbb{N} \setminus \{0\}$, then by induction $\exp(nz) = \exp(z + z + \cdots + z) = (\exp(z))^n$. If $n \in \mathbb{Z}$ and $n < 0$, then

$$\exp(nz) = \frac{1}{\exp(-nz)} = \frac{1}{(\exp z)^{-n}} = (\exp(z))^n.$$

Applying the above, we find that

$$\exp(z) = \exp(nz/n) = (\exp(z/n))^n,$$

showing that $\exp(z/n) = \sqrt[n]{\exp(z)}$ for all $n \in \mathbb{N} \setminus \{0\}$. The last statement follows by setting $z = 1$ in the above.

(d) Show that $\overline{\exp(z)} = \exp(\overline{z})$ for all $z \in \mathbb{C}$.

Solution: Since $z \mapsto \overline{z}$ is a continuous function, using the definition of the exponential function, we obtain that

$$\begin{tikzcd}
\overline{\sum_{k=0}^{n} \frac{z^k}{k!}} \arrow[d, "n \to \infty"] & = & \sum_{k=0}^{n} \frac{\overline{z}^k}{k!} \arrow[d, "n \to \infty"] \\
\overline{\exp(z)} & = & \exp(\overline{z})
\end{tikzcd}$$

(e) Show that $|\exp(it)| = 1$ for all $t \in \mathbb{R}$.

Solution: By the previous part

$$|\exp(it)|^2 = \exp(it)\overline{\exp(it)} = \exp(it)\exp(-it) = \exp(it - it) = \exp(0) = 1.$$

Challenge questions (optional)

7. Let $D \subseteq \mathbb{K}^N$ and $f, g\colon D \to \mathbb{R}$ and $x_0 \in \overline{D}$. If $D = \mathbb{R}$ or a suitable unbounded interval we also admit $x_0 = \pm\infty$. We introduce *big Oh* and *little Oh* notation to compare the asymptotic behaviour of functions as $x \to x_0$. We say

$$\begin{cases} f(x) = O(g(x)) \quad \text{as } x \to x_0 \quad \text{if } \displaystyle\limsup_{x\to x_0} \frac{|f(x)|}{|g(x)|} < \infty, \\ f(x) = o(g(x)) \quad \text{as } x \to x_0 \quad \text{if } \displaystyle\lim_{x\to x_0} \frac{|f(x)|}{|g(x)|} = 0. \end{cases}$$

(a) Let $f_1(x) = O(g(x))$ and $f_2(x) = O(g(x))$ as $x \to x_0$. Show that $\alpha f_1(x) + \beta f_2(x) = O(g(x))$ as $x \to x_0$ for all $\alpha, \beta \in \mathbb{R}$. Prove a similar statement if O is replaced by o.

Solution: Note that

$$\frac{|\alpha f_1(x) + \beta f_2(x)|}{|g(x)|} \leq |\alpha| \sup_{y\in B(x_0,r)\cap D} \frac{|f_1(y)|}{|g(y)|} + |\beta| \sup_{y\in B(x_0,r)\cap D} \frac{|f_2(y)|}{|g(y)|}$$

for all $x \in B(x_0, r) \cap D$ (if $x_0 = \infty$, then define $B(x_0, r) := (r, \infty)$). Hence

$$\sup_{x\in B(x_0,r)\cap D} \frac{|\alpha f_1(x) + \beta f_2(x)|}{|g(x)|} \leq |\alpha| \sup_{y\in B(x_0,r)\cap D} \frac{|f_1(y)|}{|g(y)|} + |\beta| \sup_{y\in B(x_0,r)\cap D} \frac{|f_2(y)|}{|g(y)|}$$

for all $r > 0$ and therefore by definition of the limit superior

$$\limsup_{x\to x_0} \frac{|\alpha f_1(x) + \beta f_2(x)|}{|g(x)|} \leq |\alpha| \limsup_{x\to x_0} \frac{|f_1(x)|}{|g(x)|} + |\beta| \limsup_{x\to x_0} \frac{|f_2(x)|}{|g(x)|} < \infty$$

as claimed. If we replace O by o, then the latter limit is zero, and so the claim about o follows with the same argument.

(b) Let $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$ as $x \to x_0$. Show that $f_1(x)f_2(x) = O\bigl(g_1(x)g_2(x)\bigr)$ as $x \to x_0$. Prove a similar statement if O is replaced by o.

Solution: Note that

$$\frac{|f_1(x)f_2(x)|}{|g_1(x)g_2(x)|} \leq \left(\sup_{y\in B(x_0,r)\cap D} \frac{|f_1(y)|}{|g_1(y)|}\right)\left(\sup_{y\in B(x_0,r)\cap D} \frac{|f_2(y)|}{|g_2(y)|}\right)$$

for all $x \in B(x_0, r) \cap D$ (if $x_0 = \infty$, then define $B(x_0, r) := (r, \infty)$). Hence

$$\sup_{x\in B(x_0,r)\cap D} \frac{|f_1(x)f_2(x)|}{|g_1(x)g_2(x)|} \leq \left(\sup_{y\in B(x_0,r)\cap D} \frac{|f_1(y)|}{|g_1(y)|}\right)\left(\sup_{y\in B(x_0,r)\cap D} \frac{|f_2(y)|}{|g_2(y)|}\right)$$

for all $r > 0$ and therefore by definition of the limit superior

$$\limsup_{x\to x_0} \frac{|f_1(x)f_2(x)|}{|g_1(x)g_2(x)|} \leq \left(\limsup_{x\to x_0} \frac{|f_1(x)|}{|g_1(x)|}\right)\left(\limsup_{x\to x_0} \frac{|f_2(x)|}{|g_2(x)|}\right) < \infty$$

as claimed. If we replace O by o, then the latter limit is zero, and so the claim about o follows with the same argument.

(c) Let $f_1(x) = O(g_1(x))$ and $f_2(x) = o(g_2(x))$ as $x \to x_0$. Show that $f_1(x)f_2(x) = o\bigl(g_1(x)g_2(x)\bigr)$ as $x \to x_0$.

Solution: This is similar to the previous part, but oberve that one of the limits on the right hand side of the last inequality is zero.

(d) Let $f(x) = O(g(x))$ as $x \to x_0$ and $\alpha > 0$. Show that $|f(x)|^\alpha = O\big(|g(x)|^\alpha\big)$ as $x \to x_0$.

Solution: Note that

$$\sup_{x \in B(x_0,r)\cap D} \frac{|f(x)|^\alpha}{|g(x)|^\alpha} = \sup_{x \in B(x_0,r)\cap D} \left|\frac{f(x)}{g(x)}\right|^\alpha = \left(\sup_{x \in B(x_0,r)\cap D} \left|\frac{f(x)}{g(x)}\right|\right)^\alpha$$

and so

$$\limsup_{x \to x_0} \frac{|f(x)|^\alpha}{|g(x)|^\alpha} = \Big(\limsup_{x \to x_0} \frac{|f(x)|}{|g(x)|}\Big)^\alpha < \infty.$$

For o note that the right-hand limit is zero.