THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 9 (Week 10)

MATH2962: Real and Complex Analysis (Advanced) Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

Questions to complete during the tutorial

1. Suppose that $f_n\colon D \to \mathbb{K}^N$ are continuous on a domain D in $\mathbb{K}^d$ and $f_n \to f$ uniformly on D.

(a) If (x_n) is a sequence in D with $x_n \to x \in D$, show that $f_n(x_n) \to f(x)$.

*(b) Let D be closed and bounded. If $f_n \to f$ pointwise on D, but not uniformly on D and $x_n \to x \in D$, can we expect that $f_n(x_n) \to f(x)$? Give a proof or counterexample.

2. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, set $f_n(x) := nx\exp(-nx^2)$.

(a) Show that f_n converges pointwise and determine the limit function.

(b) Show that f_n does not converge uniformly on any interval containing $x = 0$.

(c) Show that f_n converges uniformly on every closed interval not containing $x = 0$.

Extra questions for further practice

3. For every $n \geq 1$, we define the function $f_n(x) := |x|^{1/n}$ for $x \in \mathbb{R}$.

(a) Prove that f_n converges pointwise on $\mathbb{R}$ and determine the limit function.

(b) Show that f_n does not converge uniformly on $\mathbb{R}$.

(c) Give an interval on which f_n converges uniformly.

4. Use the binomial series to get the Taylor series expansion of the following functions about $x = 0$.

(a) $\sqrt{1+x}$; (b) $\dfrac{1}{\sqrt{1+x}}$; (c) $\sin^{-1} x$.

***5.** Suppose that $(f_n)_{n\geq 1}$ is a monotone sequence of continuous functions on $I = [a, b]$. Prove that if f_n converges pointwise on I to a continuous function f, then $f_n \to f$ uniformly on I. (This fact is known as *Dini's Theorem.*)

Challenge questions (optional)

The following guides you to a proof of the *Weierstrass approximation theorem*, a theorem asserting that all continuous functions on a closed and bounded interval can be uniformly approximated by a sequence of polynomials.

6. (a) Prove that for all $x \in [0,1]$ and all $n \in \mathbb{N} \setminus \{0\}$, we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1.$$

(b) Prove that for all $x \in [0,1]$ and all $n \in \mathbb{N} \setminus \{0\}$, we have

$$\sum_{k=1}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} = x.$$

*(c) Use (a) and (b) to prove that for all $x \in [0,1]$ and all $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k=0}^{n} \Big(\frac{k}{n} - x\Big)^2 \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{4n}. \tag{1}$$

(d) For the function $f \in C([0,1], \mathbb{R})$ define the sequence of polynomials

$$p_n(x) := \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) \binom{n}{k} x^k (1-x)^{n-k} \quad \text{for } n \in \mathbb{N} \setminus \{0\}.$$

(p_n are called *Bernstein polynomials* associated with f.) Fix $\varepsilon > 0$. By the uniform continuity (Theorem 22.8 in the lecture notes), there exists $\delta > 0$ such that

$$|f(y) - f(x)| < \frac{\epsilon}{2}$$

whenever $x, y \in [0,1]$ and $|y - x| < \delta$. For every $n \geq 1$, set

$$A_n := \{k \colon 0 \leq k \leq n,\ |x - k/n| < \delta\}, \quad B_n := \{k \colon 0 \leq k \leq n,\ |x - k/n| \geq \delta\}.$$

(i) Use (a) to show that for every $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k \in A_n} \Big| f\Big(\frac{k}{n}\Big) - f(x) \Big| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\varepsilon}{2}.$$

*(ii) Use (c) to show that for all $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k \in B_n} \Big| f\Big(\frac{k}{n}\Big) - f(x) \Big| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\|f\|_\infty}{2n\delta^2}.$$

(iii) Hence conclude that $p_n \to f$ uniformly on $[0,1]$.