

Solutions to Tutorial 9 (Week 10)

MATH2962: Real and Complex Analysis (Advanced)

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Web Page: <http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/>

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Questions marked with \* are more difficult questions.

Questions to complete during the tutorial

1. Suppose that  $f_n: D \rightarrow \mathbb{K}^N$  are continuous on a domain  $D$  in  $\mathbb{K}^d$  and  $f_n \rightarrow f$  uniformly on  $D$ .

(a) If  $(x_n)$  is a sequence in  $D$  with  $x_n \rightarrow x \in D$ , show that  $f_n(x_n) \rightarrow f(x)$ .

**Solution:** Suppose that  $x_n \rightarrow x \in D$ . We have  $f_n(x_n) \rightarrow f(x)$  if  $\|f_n(x_n) - f(x)\| \rightarrow 0$  as  $n \rightarrow \infty$ . Using the triangle inequality, we obtain that

$$\begin{aligned} \|f_n(x_n) - f(x)\| &\leq \|f_n(x_n) - f(x_n)\| + \|f(x_n) - f(x)\| \\ &\leq \sup_{y \in D} \|f_n(y) - f(y)\| + \|f(x_n) - f(x)\| \\ &= \|f_n - f\|_\infty + \|f(x_n) - f(x)\|. \end{aligned} \tag{1}$$

Since  $f_n \rightarrow f$  uniformly on  $D$ , we have  $\|f_n - f\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . Using that  $f_n$  is continuous for all  $n \in \mathbb{N}$  and  $f_n \rightarrow f$  uniformly on  $D$ , we have that  $f$  is continuous on  $D$  as well. Hence,  $f(x_n) \rightarrow f(x)$ , that is  $\|f(x_n) - f(x)\| \rightarrow 0$  as  $n \rightarrow \infty$ . By (1) and the squeeze law, we conclude that  $f_n(x_n) \rightarrow f(x)$  as  $n \rightarrow \infty$ .

\*(b) Let  $D$  be closed and bounded. If  $f_n \rightarrow f$  pointwise on  $D$ , but not uniformly on  $D$  and  $x_n \rightarrow x \in D$ , can we expect that  $f_n(x_n) \rightarrow f(x)$ ? Give a proof or counterexample.

**Solution:** The function  $f_n(x) = nx e^{-nx^2}$  converges pointwise to the zero function  $f(x) = 0$  for all  $x \in [0, 1]$ , but not uniformly on  $D = [0, 1]$ . In this case, there exist sequences  $(x_n)$  in  $D$  with  $x_n \rightarrow 0 \in D$  for which  $f_n(x_n) \not\rightarrow 0$ . Take, for example,  $x_n := 1/\sqrt{n}$  for  $n \geq 1$ . We clearly have  $x_n \rightarrow 0$  and yet

$$f_n(x_n) = \sqrt{n}/e \rightarrow \infty \text{ as } n \rightarrow \infty.$$

2. For  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ , set  $f_n(x) := nx \exp(-nx^2)$ .

(a) Show that  $f_n$  converges pointwise and determine the limit function.

**Solution:** If  $x = 0$ , then  $f_n(0) = 0$  for all  $n \in \mathbb{N}$ . Hence,  $f_n(0) \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $x \neq 0$  be fixed. Then by definition of the exponential function

$$|f_n(x)| = \frac{n|x|}{e^{nx^2}} \leq \frac{2n|x|}{(nx^2)^2} = \frac{2}{n|x|^3} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence,  $f_n \rightarrow 0$  pointwise on  $\mathbb{R}$ .

(b) Show that  $f_n$  does not converge uniformly on any interval containing  $x = 0$ .

**Solution:** Let  $I$  denote an arbitrary interval containing 0. We have  $f_n \rightarrow 0$  uniformly on  $I$  if and only if  $\|f_n\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $|f_n(y)| \leq \|f_n\|_\infty$  for every  $y \in I$ , if we can find a sequence  $x_n \in I$  with  $x_n \rightarrow 0$  such that  $|f_n(x_n)| \not\rightarrow 0$  as  $n \rightarrow \infty$ , then  $f_n$  cannot converge uniformly on  $I$ . Set  $x_n := 1/\sqrt{n}$  for  $n \geq 1$  if  $I$  contains some small interval  $[0, \nu]$  with  $\nu > 0$  (respectively,  $x_n = -1/\sqrt{n}$  if  $[-\nu, 0] \subseteq I$  for some  $\nu > 0$ ). The claim follows since  $|f_n(x_n)| = \sqrt{n}/e \rightarrow \infty$  as  $n \rightarrow \infty$ .

- (c) Show that  $f_n$  converges uniformly on every closed interval not containing  $x = 0$ .

**Solution:** Let  $J \subseteq \mathbb{R}$  be any closed interval not containing 0 where  $f_n$  converges uniformly. Then there exists  $r > 0$  such that  $|x| \geq r$  for every  $x \in J$ . Using the definition of the exponential function, we find that

$$|f_n(x)| = \frac{n|x|}{e^{nx^2}} \leq \frac{2n|x|}{(nx^2)^2} = \frac{2}{n|x|^3} \leq \frac{2}{nr^3}$$

for all  $n \in \mathbb{N} \setminus \{0\}$  and all  $x \in \mathbb{R}$  with  $|x| \geq r$ . Hence,  $f_n \rightarrow 0$  uniformly on  $J$ .

### Extra questions for further practice

3. For every  $n \geq 1$ , we define the function  $f_n(x) := |x|^{1/n}$  for  $x \in \mathbb{R}$ .

- (a) Prove that  $f_n$  converges pointwise on  $\mathbb{R}$  and determine the limit function.

**Solution:** For  $x \neq 0$ , we have  $f_n(x) = |x|^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ . Moreover,  $f_n(0) = 0$  for all  $n \geq 1$ . Hence,  $f_n$  converges pointwise to  $f$  given by  $f(x) = 1$  if  $x \neq 0$  and  $f(0) = 0$ .

- (b) Show that  $f_n$  does not converge uniformly on  $\mathbb{R}$ .

**Solution:** We know that the uniform limit of continuous functions is also continuous. Here the limit function  $f$  is not continuous at 0 although all the  $f_n$  are continuous on  $\mathbb{R}$ . Hence  $f_n$  cannot converge uniformly on  $\mathbb{R}$ .

- (c) Give an interval on which  $f_n$  converges uniformly.

**Solution:** For example,  $f_n$  converges uniformly on the interval  $[1, 2]$ . Indeed, we have

$$0 \leq f_n(x) - f(x) = f_n(x) - 1 \leq 2^{1/n} - 1 \rightarrow 0 \text{ as } n \rightarrow \infty$$

for all  $x \in [1, 2]$ .

4. Use the binomial series to get the Taylor series expansion of the following functions about  $x = 0$ .

- (a)  $\sqrt{1+x}$ ;

**Solution:** We have  $\sqrt{1+x} = (1+x)^{1/2}$ , so we apply the binomial series for  $\alpha = 1/2$ . We have  $\binom{1/2}{0} = 1$  and  $\binom{1/2}{1} = 1/2$ . Furthermore, for  $k \in \mathbb{N}$  with  $k \geq 2$ , we find that

$$\binom{1/2}{k} = \frac{1}{k!} \frac{1}{2} \left(\frac{1}{2} - 1\right) \left(\frac{1}{2} - 2\right) \dots \left(\frac{1}{2} - k + 1\right) = (-1)^{k-1} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-3)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2k)}.$$

Therefore, for  $|x| < 1$  we obtain that

$$\begin{aligned} \sqrt{1+x} &= \sum_{k=0}^{\infty} \binom{1/2}{k} x^k = 1 + \frac{1}{2}x + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{3 \cdot 5 \cdot \dots \cdot (2k-3)}{2 \cdot 4 \cdot \dots \cdot (2k)} x^k \\ &= 1 + \frac{1}{2}x - \frac{1}{2 \cdot 4}x^2 + \frac{3}{2 \cdot 4 \cdot 6}x^3 - \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \dots \end{aligned}$$

- (b)  $\frac{1}{\sqrt{1+x}}$ ;

**Solution:** We have  $1/\sqrt{1+x} = (1+x)^{-1/2}$ , so we apply the binomial series for  $\alpha = -1/2$ . We have  $\binom{-1/2}{0} = 1$  and for every  $k \in \mathbb{N} \setminus \{0\}$

$$\binom{-1/2}{k} = \frac{1}{k!} \left(-\frac{1}{2}\right) \left(-\frac{1}{2} - 1\right) \left(-\frac{1}{2} - 2\right) \dots \left(-\frac{1}{2} - k + 1\right) = (-1)^k \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2k)}.$$

Therefore, for  $|x| < 1$  we obtain that

$$\begin{aligned}\frac{1}{\sqrt{1+x}} &= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^k = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot \dots \cdot (2k)} x^k \\ &= 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 - \dots\end{aligned}$$

(c)  $\sin^{-1} x$ .

**Solution:** We know that

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \quad \text{for } |x| < 1.$$

If we substitute  $-x^2$  into the series obtained in (b), we find that

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot \dots \cdot (2k)} x^{2k} \quad \text{for } |x| < 1.$$

Integrating the series term-by-term, we infer that

$$\sin^{-1} x = c + x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot \dots \cdot (2k)} \frac{x^{2k+1}}{(2k+1)}$$

for some constant  $c$ . Since  $\sin^{-1}(0) = 0$ , we obtain that  $c = 0$ , and therefore

$$\sin^{-1} x = x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot \dots \cdot (2k)} \frac{x^{2k+1}}{(2k+1)} \quad \text{for } |x| < 1.$$

Note that the series above coincides with the Taylor series of  $\sin^{-1}$  about  $x = 0$ . Try to compute that series directly by computing the derivatives of  $\sin^{-1}$  and estimating the error terms. Doing so you will see how powerful the methods developed are!

- \*5. Suppose that  $(f_n)_{n \geq 1}$  is a monotone sequence of continuous functions on  $I = [a, b]$ . Prove that if  $f_n$  converges pointwise on  $I$  to a continuous function  $f$ , then  $f_n \rightarrow f$  uniformly on  $I$ . (This fact is known as *Dini's Theorem*.)

**Solution:** Suppose that  $(f_n)$  is a decreasing sequence of functions, that is  $f_{n+1}(x) \leq f_n(x)$  for all  $x \in [a, b]$  and all  $n \geq 1$ . Since  $f_n$  converges pointwise to  $f$  on  $I$  and  $f_n$  is decreasing, we have  $f \leq f_n$  on  $I$  for every  $n \geq 1$ . Assume by contradiction that  $f_n \not\rightarrow f$  uniformly on  $[a, b]$ . Then there exists  $\varepsilon_0 > 0$  such that for each  $k \geq 1$  there exist  $n_k \geq k$  and  $x_k \in [a, b]$  such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0. \quad (2)$$

Since  $[a, b]$  is closed and bounded, it is sequentially compact. Hence,  $(x_k)$  contains a subsequence  $(x_{k_j})$  with  $x_{k_j} \rightarrow \ell$  as  $j \rightarrow \infty$  and  $\ell \in [a, b]$ . We want to prove that

$$f_n(\ell) - f(\ell) \geq \varepsilon_0 \quad \text{for every } n \geq 1. \quad (3)$$

From  $k_j \rightarrow \infty$  as  $j \rightarrow \infty$  and  $n_{k_j} \geq k_j$ , it follows that  $n_{k_j} \rightarrow \infty$  as  $j \rightarrow \infty$ . Thus, for every  $n \geq 1$  fixed, there exists  $j_0 \geq 1$  large (with  $j_0$  depending on  $n$ ) such that  $n_{k_j} \geq n$  for all  $j \geq j_0$ . Hence, by the monotonicity of  $f_n$ ,

$$f_n(x_{k_j}) - f(x_{k_j}) \geq f_{n_{k_j}}(x_{k_j}) - f(x_{k_j}) \quad \text{for all } j \geq j_0. \quad (4)$$

On the other hand, using (2) with  $k_j$  instead of  $k$ , we have

$$f_{n_{k_j}}(x_{k_j}) - f(x_{k_j}) \geq \varepsilon_0 \quad \text{for all } j \geq j_0. \quad (5)$$

By combining (4) and (5), we obtain that

$$f_n(x_{k_j}) - f(x_{k_j}) \geq \varepsilon_0 \quad \text{for all } j \geq j_0. \quad (6)$$

Since  $x_{k_j} \rightarrow \ell$  as  $j \rightarrow \infty$ , by passing to the limit  $j \rightarrow \infty$  in (6) and using the continuity of  $f_n$  and  $f$  at  $\ell$ , we obtain (3). Since  $f_n \rightarrow f$  pointwise on  $[a, b]$  and  $\ell \in [a, b]$ , by letting  $n \rightarrow \infty$  in (3), we find that  $0 \geq \varepsilon_0$ , which is a contradiction. Hence  $f_n \rightarrow f$  uniformly on  $I = [a, b]$ .

## Challenge questions (optional)

The following guides you to a proof of the *Weierstrass approximation theorem*, a theorem asserting that all continuous functions on a closed and bounded interval can be uniformly approximated by a sequence of polynomials.

6. (a) Prove that for all  $x \in [0, 1]$  and all  $n \in \mathbb{N} \setminus \{0\}$ , we have

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = 1.$$

**Solution:** The claim follows from the binomial theorem since

$$1 = (x + (1-x))^n = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k}.$$

- (b) Prove that for all  $x \in [0, 1]$  and all  $n \in \mathbb{N} \setminus \{0\}$ , we have

$$\sum_{k=1}^n \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} = x.$$

**Solution:** Note that

$$\frac{k}{n} \binom{n}{k} = \frac{k}{n} \frac{n!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = \binom{n-1}{k-1} \quad (7)$$

Hence, if we use (a) with  $(n-1)$  rather than  $n$ , we obtain that

$$\begin{aligned} \sum_{k=1}^n \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=1}^n \binom{n-1}{k-1} x^k (1-x)^{n-k} \\ &= x \sum_{k=1}^n \binom{n-1}{k-1} x^{k-1} (1-x)^{(n-1)-(k-1)} = x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} = x. \end{aligned}$$

This proves the assertion of (b).

- \*(c) Use (a) and (b) to prove that for all  $x \in [0, 1]$  and all  $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{4n}. \quad (8)$$

**Solution:** For  $n = 1$  a simple calculation gives that the left-hand side of the inequality (8) is  $x(1-x)$ , which is clearly bounded from above by  $1/4$  for all  $x \in [0, 1]$ . We now prove (8) for  $n \geq 2$ . Note that by (7), (a) and (b), we have

$$\begin{aligned} &\sum_{k=1}^n \frac{k^2}{n^2} \binom{n}{k} x^k (1-x)^{n-k} \\ &= \sum_{k=1}^n \frac{k}{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} = x \sum_{k=0}^{n-1} \frac{k+1}{n} \binom{n-1}{k} x^k (1-x)^{n-1-k} \\ &= \frac{n-1}{n} x \sum_{k=0}^{n-1} \frac{k}{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} + \frac{x}{n} \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} \\ &= \frac{n-1}{n} x^2 + \frac{x}{n} \end{aligned}$$

for all  $x \in [0, 1]$  and all  $n \geq 2$ . Using (a), (b) and the above, we therefore find that

$$\begin{aligned} & \sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^n \frac{k^2}{n^2} \binom{n}{k} x^k (1-x)^{n-k} - 2x \sum_{k=0}^n \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} + x^2 \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \\ &= \frac{n-1}{n} x^2 + \frac{x}{n} - 2x^2 + x^2 = \frac{x(1-x)}{n} \end{aligned}$$

for all  $x \in [0, 1]$  and all  $n \geq 2$ . The required estimate follows since  $1/4$  is the maximum of  $x(1-x)$  on  $[0, 1]$ .

(d) For the function  $f \in C([0, 1], \mathbb{R})$  define the sequence of polynomials

$$p_n(x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \quad \text{for } n \in \mathbb{N} \setminus \{0\}.$$

( $p_n$  are called *Bernstein polynomials* associated with  $f$ .) Fix  $\varepsilon > 0$ . By the uniform continuity (Theorem 22.8 in the lecture notes), there exists  $\delta > 0$  such that

$$|f(y) - f(x)| < \frac{\varepsilon}{2}$$

whenever  $x, y \in [0, 1]$  and  $|y - x| < \delta$ . For every  $n \geq 1$ , set

$$A_n := \{k: 0 \leq k \leq n, |x - k/n| < \delta\}, \quad B_n := \{k: 0 \leq k \leq n, |x - k/n| \geq \delta\}.$$

(i) Use (a) to show that for every  $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k \in A_n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\varepsilon}{2}.$$

**Solution:** Using (a) and the definition of  $A_n$ , we find that

$$\sum_{k \in A_n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\varepsilon}{2} \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = \frac{\varepsilon}{2}$$

for all  $x \in [0, 1]$  and all  $n \in \mathbb{N} \setminus \{0\}$ .

\*(ii) Use (c) to show that for all  $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k \in B_n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\|f\|_\infty}{2n\delta^2}.$$

**Solution:** Since  $|k/n - x| \geq \delta$  for  $k \in B_n$ , we have

$$1 = \frac{(k/n - x)^2}{(k/n - x)^2} \leq \frac{(k/n - x)^2}{\delta^2}$$

whenever  $k \in B_n$ . Using (c), we obtain that

$$\begin{aligned} & \sum_{k \in B_n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} \leq \sum_{k \in B_n} \left( \left| f\left(\frac{k}{n}\right) \right| + |f(x)| \right) \binom{n}{k} x^k (1-x)^{n-k} \\ & \leq 2\|f\|_\infty \sum_{k \in B_n} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{2\|f\|_\infty}{\delta^2} \sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \\ & \leq \frac{2\|f\|_\infty}{\delta^2} \frac{1}{4n} = \frac{\|f\|_\infty}{2n\delta^2} \end{aligned}$$

for all  $x \in [0, 1]$  and all  $n \in \mathbb{N} \setminus \{0\}$ .

(iii) Hence conclude that  $p_n \rightarrow f$  uniformly on  $[0, 1]$ .

**Solution:** Fix  $\varepsilon > 0$  and choose  $\delta > 0$  as above. Using (a), we find that

$$\begin{aligned} |p_n(x) - f(x)| &= \left| \sum_{k=0}^n \left( f\left(\frac{k}{n}\right) - f(x) \right) \binom{n}{k} x^k (1-x)^{n-k} \right| \\ &\leq \sum_{k \in A_n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} \\ &\quad + \sum_{k \in B_n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\varepsilon}{2} + \frac{\|f\|_\infty}{2n\delta^2} \end{aligned}$$

for all  $x \in [0, 1]$  and all  $n \in \mathbb{N} \setminus \{0\}$ . Hence we can choose  $n_\varepsilon \geq 1$  such that

$$\frac{\|f\|_\infty}{2n\delta^2} < \varepsilon/2 \quad \text{for all } n > n_\varepsilon.$$

Then for all  $x \in [0, 1]$  and every  $n > n_\varepsilon$ , we have

$$|p_n(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, we conclude that  $p_n \rightarrow f$  uniformly on  $[0, 1]$ .