Questions marked with * are more difficult questions.

Material covered

(1) Uniform and pointwise convergence of sequences and series;
(2) Weierstrass $M$-test;
(3) Power series expansions.

Outcomes

This tutorial helps you to

(1) be able to distinguish uniform and pointwise convergence;
(2) work with sequences of uniformly convergent sequences of functions with applications to integration and differentiation;
(3) have a working knowledge of (real and complex) power series and their properties.

Summary of essential material

**Pointwise and Uniform Convergence.** (Partly from Tutorial 8.) Suppose $f_n, f: D \to \mathbb{K}^N$ with $D \subseteq \mathbb{K}^d$. We say that

- $f_n \to f$ pointwise on $D$ if for every (fixed) $x \in D$, we have $f_n(x) \to f(x)$ as $n \to \infty$. We call $f$ the pointwise limit of $(f_n)$.

- $f_n \to f$ uniformly on $D$ if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\|f_n(x) - f(x)\| < \varepsilon$ for all $n \geq n_\varepsilon$ and for all $x \in D$. We call $f$ the uniform limit of $(f_n)$.

**Remark.** Uniform convergence implies pointwise convergence to the same limit function. The *supremum norm* of a function $f: D \to \mathbb{K}^N$ is defined by

$$\|f\|_{\infty, D} := \sup_{x \in D} \|f(x)\|.$$

We have the following characterisation of uniform convergence:

$$f_n \to f \text{ uniformly on } D \text{ if and only if } \|f_n - f\|_{\infty, D} \to 0 \text{ as } n \to \infty.$$
As a consequence, \( f_n \to f \) uniformly on \( D \) if and only if there exists a sequence \((g_n)\) only depending on \( n \) such that \( g_n \to 0 \) as \( n \to \infty \) and

\[
\|f_n(x) - f(x)\| \leq g_n
\]

for all \( x \in D \) and all \( n \in \mathbb{N} \). It is essential that \( g_n \) does not depend on \( x \in D \)!

**Major properties of uniformly convergent sequences \((f_n)\) of continuous functions:**

(i) If \( f_n \) is continuous on \( D \) for all \( n \in \mathbb{N} \), then \( f \) is continuous on \( D \).

(ii) If \( x_n, x \in D \) are such that \( x_n \to x \), then \( f_n(x_n) \to f(x) \) as \( n \to \infty \).

As a consequence, if \( f_n \) is continuous for all \( n \in \mathbb{N} \), but the pointwise limit function \( f \) is not continuous, then the convergence is not uniform. Likewise, if \( x_n, x \in D \) with \( x_n \to x \), but \( f_n(x_n) \not\to f(x) \) as \( n \to \infty \), then the convergence is not uniform.

Regarding differentiation and integration, we have the following properties.

(i) Let \( f_n : [a, b] \to \mathbb{K} \) be continuous and \( f_n \to f \) uniformly on \( [a, b] \). Then

\[
\lim_{n \to \infty} \int_a^b f_n(t) \, dt = \int_a^b f(t) \, dt
\]

(ii) Let \( D \subseteq \mathbb{K} \) open. If \( f_n \to f \) pointwise on \( D \), and \( f'_n \to g \) locally uniformly on \( D \), then \( f \) is differentiable on \( D \) and \( f' = g \).

Let \( g_n : D \to \mathbb{K}^N \) be a sequence of functions. A series \( f(x) := \sum_{k=0}^{\infty} g_k(x) \) converges uniformly if the sequence of partial sums \( f_n(x) := \sum_{k=0}^{n} g_k(x) \) converges uniformly on \( D \). The most important test for the uniform convergence of series is the **Weierstrass M-Test**:

**Weierstrass M-Test**

If \( \sum_{k=0}^{\infty} \|g_k\|_{\infty, D} \) converges, then \( \sum_{k=0}^{\infty} g_k(x) \) converges uniformly on \( D \).

In practice this means that \( \sum_{k=0}^{\infty} g_k(x) \) converges uniformly on \( D \) if we can find \( a_k \geq 0 \) independent of \( x \in D \) such that \( \|g_k(x)\| \leq a_k \) for all \( x \in D \), and \( \sum_{k=0}^{\infty} a_k \) converges.

**Questions to complete during the tutorial**

1. Suppose that \( f_n : D \to \mathbb{K}^N \) are continuous on a domain \( D \subseteq \mathbb{K}^d \) and \( f_n \to f \) uniformly on \( D \). If \( (x_n) \) is a sequence in \( D \) with \( x_n \to x \in D \), show that \( f_n(x_n) \to f(x) \).
Solution: Suppose that \( x_n \to x \in D \). We need to show that \( \|f_n(x_n) - f(x)\| \to 0 \) as \( n \to \infty \). Using the triangle inequality, we obtain that

\[
\|f_n(x_n) - f(x)\| \leq \|f_n(x_n) - f(x_n)\| + \|f(x_n) - f(x)\|
\]

\[
\leq \sup_{y \in D} \|f_n(y) - f(y)\| + \|f(x_n) - f(x)\|
\]

\[
= \|f_n - f\|_\infty + \|f(x_n) - f(x)\|.
\]

Since \( f_n \to f \) uniformly on \( D \), we have \( \|f_n - f\|_\infty \to 0 \) as \( n \to \infty \). As \( f_n \to f \) uniformly, and \( f_n \) is continuous for all \( n \in \mathbb{N} \), the limit function \( f \) is continuous on \( D \). Hence, \( f(x_n) \to f(x) \), that is, \( \|f(x_n) - f(x)\| \to 0 \) as \( n \to \infty \). By (1) we conclude that \( f_n(x_n) \to f(x) \) as \( n \to \infty \).

2. For \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \), set \( f_n(x) := nx \exp(-nx^2) \).

(a) Show that \( f_n \) converges pointwise and determine the limit function.

Solution: If \( x \neq 0 \), then by definition of the exponential function

\[
|f_n(x)| = \frac{n|x|}{e^{nx^2}} \leq \frac{2n|x|}{(nx)^2} = \frac{2}{n|x|^3} \to 0
\]

as \( n \to \infty \). If \( x = 0 \), then \( f_n(0) = 0 \) for all \( n \in \mathbb{N} \). Hence \( f_n \to 0 \) pointwise on \( \mathbb{R} \).

(b) Show that \( f_n \) does not converge uniformly on any interval \([0, a]\) with \( a > 0 \).

Solution: We know already that \( f_n \to 0 \) pointwise. We have \( f_n \to 0 \) uniformly on \([0, a]\) if and only if \( \|f_n\|_\infty = \sup_{x \in [0, a]} |f_n(x)| \to 0 \) as \( n \to \infty \). By choosing the sequence \( 1/n \) we show that the supremum does not converge to zero. As \( 1/n \to 0 \) as \( n \to \infty \), there exists \( n_0 \in \mathbb{N} \) such that \( x_n \in [0, a] \) for all \( n > n_0 \). Hence,

\[
|f_n(1/n)| = e^{-1/n} \leq \sup_{x \in [0, a]} |f_n(x)| = \|f_n\|_\infty
\]

for all \( n > n_0 \). Since \( e^{-1/n} \to 1 \) as \( n \to \infty \), it follows that \( \|f_n\|_\infty \) cannot converge to zero as \( n \to \infty \), that is, \( (f_n) \) is not uniformly convergent on \([0, a]\) for any \( a > 0 \).

(c) Show that \( f_n \) converges uniformly on \( \mathbb{R} \setminus (-a, a) \) for every \( a > 0 \).

Solution: Suppose that \( a > 0 \) and that \( |x| \geq a \). Then by the series definition of the exponential function

\[
|f_n(x)| = \frac{n|x|}{e^{nx^2}} \leq \frac{2n|x|}{(nx)^2} = \frac{2}{n|x|^3} \leq \frac{2}{na^3}
\]

for all \( n \in \mathbb{N} \). As the term on the right hand side is independent of \(|x| \geq a \) and converges to zero, we have that \( f_n \to 0 \) uniformly with respect to \(|x| \geq a \).

3. Determine whether or not the following series converge uniformly.

(a) The geometric series \( \sum_{k=0}^{\infty} x^k \) on \((-1, 1)\);

Solution: The series does not converge uniformly. For \( x \in (-1, 1) \) and \( n \in \mathbb{N} \) we set

\[
f(x) := \sum_{k=0}^{\infty} x^k \quad \text{and} \quad f_n(x) := \sum_{k=0}^{n} x^k.
\]
Uniform convergence means that \( \|f_n - f\|_\infty \to 0 \). Using the formula for the partial sum of a geometric series we get
\[
|f_n(x) - f(x)| = \left| \sum_{k=0}^{n} x^k - \sum_{k=n+1}^{\infty} x^k \right| = \left| \sum_{k=n+1}^{\infty} x^k \right| = \left| \frac{x^{n+1}}{1-x} \right| = \left| \frac{|x|^{n+1}}{1-x} \right|.
\]
We have that \( \frac{|x|^{n+1}}{1-x} \to \infty \) as \( x \to 1 \), and hence,
\[
\|f_n - f\|_\infty = \sup_{x \in (-1, 1)} \left| \frac{|x|^{n+1}}{1-x} \right| = \infty
\]
for all \( n \in \mathbb{N} \). Hence, \( f_n \) cannot converge uniformly on \((-1, 1)\).

(b) \( F(x) := \sum_{k=0}^{\infty} 2^{-k} \cos(k \pi x) \) for \( x \in \mathbb{R} \). Show that the function \( F \) is differentiable;

**Solution:** We clearly have \( |2^{-k} \cos(k \pi x)| \leq 2^{-k} \). The geometric series \( \sum_{k=0}^{\infty} \frac{1}{2^k} \) converges. Hence, the Weierstrass M-Test implies uniform convergence of the given series. Differentiating term by term, we get the series
\[
-\pi \sum_{k=0}^{\infty} k2^{-k} \sin(k \pi x).
\]

As the series \( \sum_{k=0}^{\infty} k2^{-k} \) converges, the Weierstrass M-Test implies the uniform convergence of \( (*) \) converges uniformly. Hence, \( F \) is differentiable.

(c) \( \sum_{n=0}^{\infty} \frac{x^2}{(1 + x^2)^n} \) for \( x \in \mathbb{R} \) (you can compute the limit using the geometric series);

**Solution:** If \( x \neq 0 \), using the formula for the geometric series, we get
\[
\sum_{n=0}^{\infty} \frac{x^2}{(1 + x^2)^n} = x^2 \sum_{n=0}^{\infty} \frac{1}{(1 + x^2)^n} = \frac{x^2}{1 - \frac{1}{1+x^2}} = \frac{x^2(1+x^2)}{1 + x^2 - 1} = 1 + x^2.
\]
If \( x = 0 \), then all terms in the series are zero, so the limit is 0. Hence,
\[
\sum_{n=0}^{\infty} \frac{x^2}{(1 + x^2)^n} = \begin{cases} 1 + x^2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}
\]
The limit function is not continuous, but all the partial sums of the series are continuous. Hence, the series does not converge uniformly.

*(d)* \( \sum_{n=1}^{\infty} \frac{x}{n(1 + nx^2)} \) for \( x \in \mathbb{R} \);

**Solution:** We determine an upper bound for the absolute values of the terms \( g_n(x) := \frac{x}{n(1 + nx^2)} \). From the elementary inequality \( 2ab \leq a^2 + b^2 \), we get \( (1 + nx^2) \geq 2\sqrt{n}|x| \) for all \( x \in \mathbb{R} \). Hence, for all \( x \in \mathbb{R} \setminus \{0\} \), we have
\[
|g_n(x)| \leq \frac{|x|}{2n\sqrt{n}|x|} = \frac{1}{2n^{3/2}}.
\]
This inequality holds for all \( x \in \mathbb{R} \) (since \( g_n(0) = 0 \) for all \( n \geq 1 \)). Thus,

\[
\|g_n\|_\infty \leq \frac{1}{2n^{3/2}}.
\]

Since \( 3/2 > 1 \) the series \( \sum_{n=0}^{\infty} \frac{1}{n^{3/2}} \) converges and the Weierstrass M-Test implies the uniform convergence of the original series. Alternatively, we could determine the maxima of the terms \( g_n(x) \) and check whether the corresponding series converges.

4. This question is an application of properties of uniformly convergent sequences to prove the existence and uniqueness of a solution to a system of ordinary differential equations (omitted from almost all courses on ordinary differential equations).

Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous, that is, there exists \( L > 0 \) such that \( |f(x) - f(y)| \leq L|x - y| \) for all \( x, y \in \mathbb{R} \). Consider the differential equation

\[
u'(t) = f(u(t)), \quad u(0) = x_0.
\]

The aim of this question is to demonstrate how the theory from lectures can be applied to prove the existence of a solution to the above differential equation on an interval \( [-T, T] \).

(a) Show that \( u \in C^1([-T, T], \mathbb{R}) \) is a solution to (2) if and only if for all \( t \in [-T, T] \)

\[
u(t) = x_0 + \int_0^t f(u(s)) \, ds.
\]

**Solution:** This is a simple consequence of the fundamental theorem of calculus.

(b) Define a sequence of functions \( u_n \) inductively by \( u_0(t) := x_0 \) and

\[
u_{n+1}(t) := x_0 + \int_0^t f(u_n(s)) \, ds.
\]

Prove that for every \( T > 0 \) and \( n \in \mathbb{N} \) \( \sup_{|t| \leq T} |u_{n+1}(t) - u_n(t)| \leq (LT)^n T |f(x_0)| \).

**Solution:** We do an induction by \( n \). If \( n = 0 \) we have

\[
|u_1(t) - u_0(t)| = \left| \int_0^t f(x_0) \, ds \right| \leq |t| |f(x_0)| \leq T |f(x_0)| = (LT)^0 T |f(x_0)|,
\]

so \( \sup_{|t| \leq T} |u_1(t) - u_0(t)| \leq (LT)^0 T |f(x_0)| \). Now assume the inequality holds for some \( n \geq 0 \). If \( t \in [0, T] \), then by assumption on \( f \) and the induction hypothesis

\[
|u_{n+2}(t) - u_{n+1}(t)| = \left| \int_0^t f(u_{n+1}(s)) - f(u_n(s)) \, ds \right|
\]

\[
\leq \int_0^t |f(u_{n+1}(s)) - f(u_n(s))| \, ds \leq L \int_0^t |u_{n+1}(s) - u_n(s)| \, ds
\]

\[
\leq L \int_0^t \sup_{s \in [-T,T]} |u_{n+1}(s) - u_n(s)| \, ds \leq LT \sup_{s \in [-T,T]} |u_{n+1}(s) - u_n(s)|
\]

\[
\leq (LT)^{n+1} T |f(x_0)|
\]

A similar argument applies if \( t \in [-T, 0] \). Taking the supremum over \( t \in [-T, T] \) on the left hand side the claim follows.
*(c) Choose $T < L^{-1}$ and show that $(u_n)$ is uniformly Cauchy on $[-T, T]$.

**Solution:** We set $\|g\|_\infty := \sup_{t \in [-T, T]} |g(t)|$. If $m \geq n$ then by part (b) we have

$$\|u_m - u_n\|_\infty \leq \sum_{k=n}^{m-1} \|u_{k+1} - u_k\|_\infty \leq T|f(x_0)| \sum_{k=n}^{m-1} (LT)^k$$

$$= T|f(x_0)|(LT)^n \sum_{k=0}^{m-n-1} (LT)^k$$

$$= T|f(x_0)|(LT)^n \frac{1 - (LT)^{m-n}}{1 - LT}$$

$$= \frac{T|f(x_0)|}{1 - LT} \left( (LT)^n - (LT)^m \right).$$

If $T < L^{-1}$, then $LT < 1$, so $(LT)^n \to 0$ as $n \to \infty$. Hence $(LT)^n$ is a Cauchy sequence and therefore, given $\varepsilon > 0$ arbitrary, there exists $n_0 \in \mathbb{N}$ such that

$$|(LT)^n - (LT)^m| \leq \frac{1 - LT}{T|f(x_0)|} \varepsilon$$

for all $n, m > n_0$. Hence from the above

$$\|u_m - u_n\|_\infty \leq \frac{T|f(x_0)|}{1 - LT} \frac{1 - LT}{T|f(x_0)|} \varepsilon = \varepsilon$$

for all $n, m > n_0$, showing that $(u_n)$ is uniformly Cauchy on $[-T, T]$.

*(d) Let $T < L^{-1}$. Using the fact that $f$ is Lipschitz, show that $f(u_n(t)) \to f(u(t))$ uniformly with respect to $t \in [-T, T]$.

**Solution:** If we choose $T < L^{-1}$, then by (c), the sequence $(u_n)$ is uniformly Cauchy on $[-T, T]$. By a result in lectures $u_n \to u$ uniformly on $[-T, T]$. By the assumption on $f$

$$|f(u_n(t)) - f(u(t))| \leq L|u_n(t) - u(t)| \leq L\|u_n - u\|_\infty,$$

for all $t \in [-T, T]$. Hence $\|f \circ u_n - f \circ u\|_\infty \leq L\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Hence, $f(u_n(t)) \to f(u(t))$ uniformly with respect to $t \in [-T, T]$.

(e) Show that there exists $T > 0$ such that (2) has a solution on $[-T, T]$.

**Solution:** If we choose $T < L^{-1}$, then by (d) $f(u_n(t)) \to f(u(t))$ uniformly with respect to $t \in [-T, T]$ Hence we can take a limit under the integral sign and get

$$u_{n+1}(t) = x_0 + \int_0^t f(u_n(s)) ds \downarrow_{n \to \infty} u(t) = x_0 + \int_0^t f(u(s)) ds.$$ 

By part (a) the function $u$ is a solution to (2) on $[-T, T]$. 
Extra questions for further practice

5. For every \( n \geq 1 \), we define the function \( f_n(x) := |x|^{1/n} \) for \( x \in \mathbb{R} \).

(a) Prove that \( f_n \) converges pointwise on \( \mathbb{R} \) and determine the limit function.

**Solution:** For \( x \neq 0 \), we have \( f_n(x) = |x|^{1/n} \to 1 \) as \( n \to \infty \). Moreover, \( f_n(0) = 0 \) for all \( n \geq 1 \). Hence, \( f_n \) converges pointwise to \( f \) given by \( f(x) = 1 \) if \( x \neq 0 \) and \( f(0) = 0 \).

(b) Show that \( f_n \) does not converge uniformly on \( \mathbb{R} \).

**Solution:** We know that the uniform limit of continuous functions is also continuous. Here the limit function \( f \) is not continuous at 0 although all the \( f_n \) are continuous on \( \mathbb{R} \). Hence \( f_n \) cannot converge uniformly on \( \mathbb{R} \).

(c) Give an interval on which \( f_n \) converges uniformly.

**Solution:** For example, \( f_n \) converges uniformly on the interval \([1, 2]\). Indeed, we have

\[
0 \leq f_n(x) - f(x) = f_n(x) - 1 \leq 2^{1/n} - 1 \to 0
\]
as \( n \to \infty \) for all \( x \in [1, 2] \). As \( 2^{1/n} - 1 \) is independent of \( x \in [1, 2] \), we conclude that \( f_n \to 1 \) uniformly on \([1, 2]\).

6. Use the binomial series to get the Taylor series expansion of the following functions about \( x = 0 \).

(a) \( \sqrt{1+x} \):

**Solution:** We have \( \sqrt{1+x} = (1+x)^{1/2} \), so we apply the binomial series for \( \alpha = 1/2 \). We have \((\frac{1}{2})_0 = 1\) and \((\frac{1}{2})_1 = 1/2\). Furthermore, for \( k \in \mathbb{N} \) with \( k \geq 2 \), we find that

\[
\left(\frac{1}{2}\right) k! \left(\frac{1}{2} - 1\right) \left(\frac{1}{2} - 2\right) \cdots \left(\frac{1}{2} - k + 1\right) = (-1)^{k-1} \frac{1 \cdot 3 \cdot 5 \cdots (2k-3)}{2 \cdot 4 \cdot 6 \cdots (2k)}.
\]

Therefore, for \(|x| < 1\) we obtain that

\[
\sqrt{1+x} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right) k! \left(\frac{1}{2} - \frac{1}{2} - 1\right) \left(\frac{1}{2} - \frac{1}{2} - 2\right) \cdots \left(\frac{1}{2} - \frac{1}{2} - k + 1\right) = 1 + 1 \left(\frac{1}{2}\right) x + \frac{1 \cdot 3 \cdot 5 \cdots (2k-3)}{2 \cdot 4 \cdot 6 \cdots (2k)} x^k = 1 + \frac{1}{2} x - \frac{1}{2 \cdot 4} x^2 + \frac{3}{2 \cdot 4 \cdot 6} x^3 - \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} x^4 + \ldots
\]

(b) \( \frac{1}{\sqrt{1+x}} \):

**Solution:** We have \( 1/\sqrt{1+x} = (1+x)^{-1/2} \), so we apply the binomial series for \( \alpha = -1/2 \). We have \((-\frac{1}{2})_0 = 1\) and for every \( k \in \mathbb{N} \setminus \{0\}\)

\[
\left(-\frac{1}{2}\right) k! \left(-\frac{1}{2} - 1\right) \left(-\frac{1}{2} - 2\right) \cdots \left(-\frac{1}{2} - k + 1\right) = (-1)^{k-1} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)}.
\]

Therefore, for \(|x| < 1\) we obtain that

\[
\frac{1}{\sqrt{1+x}} = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right) k! \left(-\frac{1}{2} - \frac{1}{2} - 1\right) \left(-\frac{1}{2} - \frac{1}{2} - 2\right) \cdots \left(-\frac{1}{2} - \frac{1}{2} - k + 1\right) = 1 - 1 \left(\frac{1}{2}\right) x + \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)} x^k = 1 - \frac{1}{2} x + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^2 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} x^3 - \ldots
\]
(c) \( \sin^{-1} x \).

Solution: We know that for \(|x| < 1\)

\[
(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}.
\]

If we substitute \(-x^2\) into the series obtained in (b), we find that for \(|x| < 1\)

\[
(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdots (2k)} \frac{x^{2k}}{2k+1}.
\]

Integrating the series term-by-term we get

\[
\sin^{-1} x = c + x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdots (2k) \cdot 2k+1} x^{2k+1}
\]

for some constant \(c\). Since \(\sin^{-1}(0) = 0\) we have that \(c = 0\), and therefore

\[
\sin^{-1} x = x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdots (2k) \cdot 2k+1} x^{2k+1}
\]

for \(|x| < 1\). Note that the series above coincides with the Taylor series of \(\sin^{-1}\) about \(x = 0\). Try to compute that series directly by computing the derivatives of \(\sin^{-1}\) and estimating the error terms. Doing so you will see how powerful the methods developed are!

*7. Suppose that \(f_n \to f\) pointwise, that \(f_n, f\) are continuous and that \(f_{n+1}(x) \leq f_n(x)\) for all \(x \in [a,b]\) and all \(n \in \mathbb{N}\). Prove that \(f_n \to f\) uniformly. (This fact is known as Dini’s theorem.)

Solution: Suppose that \(f_n\) does not converge to \(f\) uniformly. Then there exists \(\varepsilon > 0\) such that for each \(k \in \mathbb{N}\) there exists \(n_k \geq k\) and \(x_k \in [a,b]\) such that

\[
|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon.
\]

By the Bolzano-Weierstrass theorem and since \([a, b]\) is closed and bounded there exists a subsequence \((x_{k_j})\) with \(x_{k_j} \to x\) such that \(x \in [a,b]\). Since \(f_n\) is decreasing we have \(f \leq f_n\) and

\[
f_n(x_k) - f(x_k) \geq f_{n_k}(x_k) - f(x_k) \geq \varepsilon
\]

for all \(k \in \mathbb{N}\) with \(n_k \geq n\). Now by continuity of \(f_n\) and \(f\) we have

\[
\lim_{j \to \infty} (f_n(x_{k_j}) - f(x_{k_j})) = f_n(x) - f(x) \geq \varepsilon.
\]

The above works for any choice of \(n \in \mathbb{N}\) and therefore \(f_n(x) - f(x) \geq \varepsilon > 0\) for all \(n \in \mathbb{N}\). Since \(f_n \to f\) pointwise \(f_n(x) - f(x) \to 0\) which is a contradiction. Hence \(f_n \to f\) uniformly.
Challenge questions (optional)

The following guides you to a proof of the Weierstrass approximation theorem, a theorem asserting that all continuous functions on a closed and bounded interval can be uniformly approximated by a sequence of polynomials.

8. (a) Use the binomial theorem to prove that

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$$

for all $x \in [0, 1]$ and all $n \in \mathbb{N}$.

**Solution:** By the binomial theorem

$$1 = (x + (1-x))^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

as claimed.

(b) Hence, prove the identities

$$\sum_{k=1}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} = x$$

and

$$\sum_{k=1}^{n} \frac{k^2}{n^2} \binom{n}{k} x^k (1-x)^{n-k} = \frac{n-1}{n} x^2 + \frac{x}{n}$$

for all $x \in [0, 1]$ and all $n \in \mathbb{N}$.

**Solution:** Note that by a standard identity for the binomial coefficients

$$\frac{k}{n} \binom{n}{k} = \frac{k}{n} \binom{n}{k} = \frac{(n-1)!}{(k-1)!(n-1)-(k-1)!} = \frac{(n-1)}{k-1}$$

Hence, if we use (a) with $(n-1)$ rather than $n$, we obtain that

$$\sum_{k=1}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=1}^{n} \frac{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= x \sum_{k=1}^{n} \frac{n-1}{k-1} x^{k-1} (1-x)^{(n-1)-(k-1)} = x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} = x$$

as claimed. Similarly, we get

$$\sum_{k=1}^{n} \frac{k^2}{n^2} \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{k}{n} \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= x \sum_{k=0}^{n-1} \frac{k + 1}{n} \binom{n-1}{k} x^k (1-x)^{n-1-k}$$

$$= \frac{n-1}{n} x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} + x \sum_{k=0}^{n-1} \binom{n}{k} x^k (1-x)^{n-1-k}$$

$$= \frac{n-1}{n} x^2 + \frac{x}{n}$$
(c) Use parts (a) and (b) to prove that
\[
\sum_{k=0}^{n} \left( \frac{k}{n} - x \right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n} \leq \frac{1}{4n}
\]
for all \(x \in [0, 1]\) and all \(n \geq 1\).

**Solution:** Using (a) and (b), we find that
\[
\sum_{k=0}^{n} \left( \frac{k}{n} - x \right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} \left( \frac{k}{n} \right)^2 \binom{n}{k} x^k (1-x)^{n-k} - 2x \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k}
\]
\[
+ x^2 \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \frac{n-1}{n} x^2 + \frac{x}{n} - 2x^2 + x^2 = \frac{x(1-x)}{n}
\]
for all \(x \in [0, 1]\) and all \(n \geq 1\). The required estimate follows since 1/4 is the maximum of the quadratic \(x(1-x)\).

(d) For the function \(f \in C([0, 1], \mathbb{R})\) define the sequence of polynomials
\[
p_n(x) := \sum_{k=0}^{n} f\left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k}.
\]
(\(p_n\) are called *Bernstein polynomials* associated with \(f\).) Fix \(\varepsilon > 0\). You are given that \(f\) is uniformly continuous, that is, for every \(\varepsilon > 0\), there exists \(\delta > 0\) so that
\[
|f(y) - f(x)| < \frac{\varepsilon}{2}
\]
whenever \(x, y \in [0, 1]\) and \(|y-x| < \delta\). For every \(n \geq 1\), set
\[
A_n := \{k \in \mathbb{N} : 0 \leq k \leq n, |x - k/n| < \delta\},
\]
\[
B_n := \{k \in \mathbb{N} : 0 \leq k \leq n, |x - k/n| \geq \delta\}.
\]

(i) Use (a) to show that
\[
\sum_{k \in A_n} \left| f\left( \frac{k}{n} \right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\varepsilon}{2}
\]
for all \(n \in \mathbb{N}\).

**Solution:** Using (a) and the definition of \(A_n\)
\[
\sum_{k \in A_n} \left| f\left( \frac{k}{n} \right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\varepsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \frac{\varepsilon}{2}
\]
for all \(x \in [0, 1]\) and all \(n \in \mathbb{N} \setminus \{0\}\).
**(ii)** Use (c) to show that

\[ \sum_{k \in B_n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1 - x)^{n-k} < \frac{\|f\|_{\infty}}{2n\delta^2} \]

for all \( n \in \mathbb{N} \).

**Solution:** Since \( |k/n - x| \geq \delta \) for \( k \in B_n \), we have

\[ 1 = \frac{(k/n - x)^2}{(k/n - x)^2} \leq \frac{(k/n - x)^2}{\delta^2} \]

whenever \( k \in B_n \). Using (c), we get

\[ \sum_{k \in B_n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1 - x)^{n-k} \leq 2\|f\|_{\infty} \sum_{k \in B_n} \binom{n}{k} x^k (1 - x)^{n-k} \]

\[ \leq \frac{2\|f\|_{\infty}}{\delta^2} \sum_{k=0}^{n} \left( \frac{k}{n} - x \right)^2 \binom{n}{k} x^k (1 - x)^{n-k} \]

\[ \leq \frac{2\|f\|_{\infty}}{\delta^2} \sum_{k=0}^{n} \frac{1}{4n} \]

\[ = \frac{\|f\|_{\infty}}{2n\delta^2} \]

for all \( x \in [0,1] \) and all \( n \in \mathbb{N} \).

(iii) Hence conclude that \( p_n \to f \) uniformly on \([0,1]\).

**Solution:** Fix \( \varepsilon > 0 \) and choose \( \delta > 0 \) as above. Using (a), we find that

\[ |p_n(x) - f(x)| = \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1 - x)^{n-k} \]

\[ \leq \sum_{k \in A_n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1 - x)^{n-k} \]

\[ + \sum_{k \in B_n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1 - x)^{n-k} \]

\[ < \frac{\varepsilon}{2} + \frac{\|f\|_{\infty}}{2n\delta^2} \]

for all \( x \in [0,1] \) and all \( n \in \mathbb{N} \setminus \{0\} \). Hence we can choose \( n_\varepsilon \geq 1 \) such that

\[ \frac{\|f\|_{\infty}}{2n\delta^2} < \varepsilon/2 \]

for all \( n > n_0 \). Then

\[ |p_n(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

for all \( x \in [0,1] \) and all \( n > n_0 \). Since the above arguments work for every choice of \( \varepsilon > 0 \), we conclude that \( p_n \to f \) uniformly.

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