The University of Sydney
School of Mathematics and Statistics

# Tutorial 10 (Week 11)

MATH2962: Real and Complex Analysis (Advanced) — Semester 1, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/IM/MATH2962/
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Questions marked with * are more difficult questions.

## Questions to complete during the tutorial

**1.** Consider the function $f(x) = |x|^2$ for $x \in \mathbb{C}$.

(a) Show that $f$ is differentiable as a function on $\mathbb{R}$ with derivative $f'(x) = 2x$.

(b) Show that $f$ is not differentiable as a function on $\mathbb{C}$.

**2.** Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ by $\varphi(x) := \exp(-1/x)$ if $x > 0$ and $\varphi(x) := 0$ for $x \leq 0$.

(a) Prove by induction that for $x > 0$, the $n$-th derivative is given by $\varphi^{(n)}(x) = p_n(1/x)\varphi(x)$, where $p_n$ is a polynomial.

(b) Use (a) and induction to show that $\varphi^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

(c) Use the above to show that $\varphi$ is not analytic even though $\varphi$ has derivatives of all orders.

**3.** For $z \in \mathbb{C}$ define

$$\cos z := \frac{\exp(iz) + \exp(-iz)}{2}, \qquad \sin z := \frac{\exp(iz) - \exp(-iz)}{2i}$$

and the hyperbolic functions

$$\cosh z := \frac{\exp(z) + \exp(-z)}{2}, \qquad \sinh z := \frac{\exp(z) - \exp(-z)}{2}.$$

If $z \in \mathbb{R}$ one can show that the above functions are the well known trigonometric and hyperbolic functions.

(a) Use the exponential series to write down power series expansions for the above functions.

(b) Show that $\sin' z = \cos z$, $\cos' z = -\sin z$, $\sinh' z = \cosh z$ and $\cosh' z = \sinh z$.

(c) Show that $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ and $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$;

(d) Show that $\sinh(iz) = i \sin z$, $\cosh(iz) = \cos z$, $\sin(iz) = i \sinh z$ and $\cos(iz) = \cosh z$. Derive expressions for real and imaginary parts of the given functions.

## Extra questions for further practice

**4.** Suppose that $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is a complex power series with radius of convergence $\varrho > 0$.

(a) The vector $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$ is called a *multi-index* (all components are non-negative integers). We call $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_k$ the length of the multi-index $\alpha$. Show that

$$\big(f(z)\big)^k = \sum_{n=0}^{\infty} \Big( \sum_{|\alpha|=n} a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_k} \Big) z^n$$

for all $k \in \mathbb{N} \setminus \{0\}$ and all $z \in \mathbb{C}$ with $|z| < \varrho$.
*Hint:* Use induction by $k$ and Cauchy products.

(b) Let $g(z) = \sum_{j=1}^{\infty} b_j z^j$ be a complex power series with zero constant term. Show that $f(g(z))$ has a power series representation about $z = 0$ as well.
*Hint:* Use the double series theorem.

*(c) Use (a) and (b) to show that $1/f(z)$ has a power series representation about $z = 0$ if $f(0) = a_0 \neq 0$.
*Hint:* Use the representation $\dfrac{1}{f(z)} = \dfrac{1}{a_0 + \sum_{j=1}^{\infty} a_j z^j}$ and the formula for the sum of a geometric series.

**5.** Let $\varphi \colon \mathbb{R} \to \mathbb{R}$ be the function considered in Question **2** and define $\psi(x) := c\varphi(1 - |x|^2)$, where $c$ is a constant such that

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 1.$$

Moreover, for $n \in \mathbb{N} \setminus \{0\}$, let $\psi_n(x) := n\psi(nx)$ for all $x \in \mathbb{R}$.

(a) Show that $\psi_n$ has derivatives of all orders, and that $\psi_n(x) > 0$ if $|x| < 1/n$ and that $\psi_n(x) = 0$ if $|x| \geq 1/n$. Argue why such functions do not exist on $\mathbb{C}$.

(b) Show that for all $n \in \mathbb{N} \setminus \{0\}$, we have

$$\int_{-\infty}^{\infty} \psi_n(x)\, dx = 1.$$

(c) Let $f \in C(\mathbb{R}, \mathbb{R})$. For $x \in \mathbb{R}$ define the parameter integrals

$$f_n(x) := \int_{-\infty}^{\infty} \psi_n(y) f(x - y)\, dy$$

for all $n \in \mathbb{N} \setminus \{0\}$. Show that $f_n$ has derivatives of all orders and that $f_n \to f$ locally uniformly on $\mathbb{R}$. (The family $(\psi_n)$ is called a *mollifier* or *approximate identity* and used to approximate functions by very smooth functions.)

## Challenge questions (optional)

The following question is an application of properties of uniformly convergent sequences to prove the existence and uniqueness of a solution to a system of ordinary differential equations (omitted from almost all courses on ordinary differential equations).

***6.** Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, that is, there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Consider the differential equation

$$u'(t) = f(u(t)), \qquad u(0) = x_0. \tag{2}$$

The aim of this question is to demonstrate how the theory from lectures can be applied to prove the existence of a solution to the above differential equation on some interval $[-T, T]$.

(a) Show that $u \in C^1([-T, T], \mathbb{R})$ is a solution to (2) if and only if for all $t \in [-T, T]$

$$u(t) = x_0 + \int_0^t f(u(s))\, ds. \tag{3}$$

(b) Define a sequence of functions $u_n$ inductively by $u_0(t) := x_0$ and

$$u_{n+1}(t) := x_0 + \int_0^t f(u_n(s))\, ds \quad \text{for } n \in \mathbb{N}.$$

Prove that for every $T > 0$ and $n \in \mathbb{N}$, we have

$$\sup_{|t| \leq T} |u_{n+1}(t) - u_n(t)| \leq (LT)^n T |f(x_0)|. \tag{4}$$

(c) Choose $T < L^{-1}$ and show that $(u_n)$ is uniformly Cauchy on $[-T, T]$.

(d) Use the above to show that there exists $T > 0$ such that (2) has a solution on $[-T, T]$.

The following question guides you through a rigorous construction of the Riemann integral for continuous functions. The essential ingredient is that continuous functions are *uniformly* continuous on compact sets! This is optional material for the interested student, and by no means expected that you work through this question!

***7.** We suppose that $I = [a, b]$ is a compact interval and $f\colon I \to \mathbb{R}$ is bounded. By a partition of $I$, we mean a set $P = \{x_i\colon i = 1, \dots, n\}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. If $\xi_i \in [x_{i-1}, x_i]$ and $\Delta x_i := x_i - x_{i-1}$ then

$$\sum_{k=1}^{n} f(\xi_i)\Delta x_i$$

is called a *Riemann sum* for $f$.

(a) Set $m_i := \inf\{f(x)\colon x \in [x_{i-1}, x_i]\}$ and $M_i := \sup\{f(x)\colon x \in [x_{i-1}, x_i]\}$. Then set

$$L_f(P) := \sum_{k=1}^{n} m_i \Delta x_i \qquad \text{and} \qquad U_f(P) := \sum_{k=1}^{n} M_i \Delta x_i.$$

$U_f(P)$ and $L_f(P)$ are called the *upper* and *lower Riemann sums* of $f$ for the partition $P$. Show that $U_f(P)$ and $L_f(P)$ exist for every partition $P$, and that

$$L_f(P) \leq \sum_{k=1}^{n} f(\xi_i)\Delta x_i \leq U_f(P)$$

for every partition $P$ of $I$ and every choice of $\xi_i \in [x_{i-1}, x_i]$.

(b) For every bounded function $f\colon I \to \mathbb{R}$, we set

$$\begin{aligned} L(f) &:= \sup\{L_f(P)\colon P \text{ is a partition of } I\}, \\ U(f) &:= \inf\{U_f(P)\colon P \text{ is a partition of } I\}. \end{aligned}$$

Show that both quantities are finite and that $L(f) \leq U(f)$.

(c) A function is called *Riemann integrable* if $L(f) = U(f)$. If that is the case, we write

$$\int_a^b f(x)\,dx := L(f) = U(f).$$

Show that $f$ is Riemann integrable if and only if for every $\varepsilon > 0$ there exists a partition $P$ such that

$$U_f(P) - L_f(P) < \varepsilon.$$

(d) Show that $f$ is Riemann integrable if $f \in C([a, b], \mathbb{R})$. To this aim, use that continuous functions are uniformly continuous on compact sets.