

Exercise 2 Solutions

1. (a) The parameter of interest is μ , the average hourly wage in a given company. The objective is to determine whether this company pays inferior wages in comparison to the total industry. The hypothesis to be tested is

$$H_0 : \mu = 21.6 \quad \text{vs.} \quad H_1 : \mu < 21.6.$$

- (b) The test statistic is $t = \sqrt{n}(\bar{X} - 21.6)/2.5$. Using the information contained in the sample, we have

$$t = \sqrt{40}(20.3 - 21.6)/2.5 = -3.29, \quad p = \Phi(t) = 0.0005.$$

Hence, there is very strong evidence that this company is paying substandard wages.

- (c) We reject the null hypothesis H_0 at the level $\alpha = 0.01$ if

$$\sqrt{40}(\bar{x} - 21.6)/2.5 \leq -z_{0.01} = -2.32,$$

that is, we reject H_0 at the level $\alpha = 0.01$ if $\bar{x} < 20.68$.

2. (a) For the z -test, we should reject the null hypothesis H_0 if

$$\frac{\sqrt{n}|\bar{X} - \mu_0|}{\sigma} \geq z_{\frac{\alpha}{2}}.$$

With the information contained in the question ($n = 25, \sigma = 5, \mu_0 = 7$), we reject the H_0 if

$$|\bar{X} - 7| \geq z_{0.025} = 1.96,$$

The rejection region based on the \bar{X} is given by

$$\bar{X} \geq 7 + 1.96 = 8.96 \quad \cup \quad \bar{X} \leq 7 - 1.96 = 5.04.$$

- (b) Recall the power is the probability of rejecting H_0 with the true value $\mu = \mu_1$, i.e.,

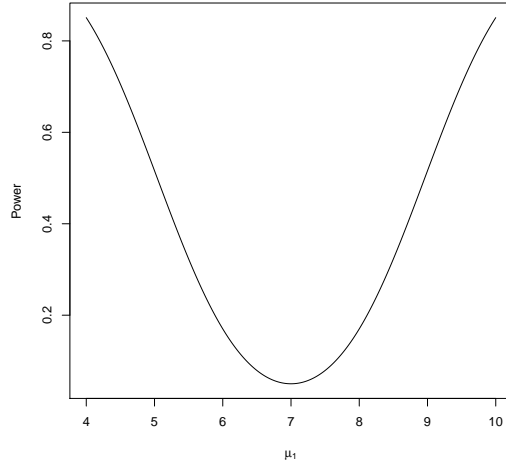
$$\begin{aligned} \text{Power}(\mu_1) &= \mathbf{P}_{\mu_1}(\bar{X} \geq 8.96) + \mathbf{P}_{\mu_1}(\bar{X} \leq 5.04) \\ &= \mathbf{P}_{\mu_1}(\bar{X} - \mu_1 \geq 8.96 - \mu_1) + \mathbf{P}_{\mu_1}(\bar{X} - \mu_1 \leq 5.04 - \mu_1) \\ &= 1 - \Phi(8.96 - \mu_1) + \Phi(5.04 - \mu_1), \end{aligned}$$

where we have used the result: $\sqrt{n}(\bar{X} - \mu)/\sigma \sim N(0, 1)$.

With $\mu_1 = 5, 6, 6.5, 7, 7.5, 8, 9$, the corresponding powers are

$$\begin{aligned} \text{Power}(5) &= 0.516, & \text{Power}(6) &= 0.17, & \text{Power}(6.5) &= 0.079, \\ \text{Power}(7) &= 0.05, & \text{Power}(7.5) &= 0.079, & \text{Power}(8) &= 0.17, & \text{Power}(9) &= 0.516. \end{aligned}$$

The graph of the power function is as follows:



3. Let \bar{X} be the sample average that generates \bar{x} . By the central limit theorem, for $n \geq 20$,

$$\frac{\sqrt{n}(\bar{X} - \beta)}{\beta} \sim N(0, 1) \quad \text{approximately.}$$

Let $p = \mathbf{P}_{H_0}(\bar{X} \geq c)$. It follows from the central limit theorem that

$$p = \mathbf{P}_{H_0}\left(\sqrt{25}(\bar{X} - 1) \geq \sqrt{25}(c - 1)\right) \approx 1 - \Phi(5(c - 1)).$$

Choose c such that

$$1 - \Phi(5(c - 1)) = 0.05.$$

Then the test that rejects H_0 if $\bar{x} \geq c$ has significance level $\alpha = 0.05$.

The c is given by

$$5(c - 1) = z_{0.05} = 1.645, \quad c = 1 + 1.645/5 = 1.329.$$

4. First we find the likelihood function

$$L(\sigma|\mathbf{x}) = \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_i^2}{2\sigma^2}} \right) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2}.$$

Now, by the Neyman-Pearson lemma, the most powerful test for testing $H_0 : \sigma = \sigma_0$ vs. $H_1 : \sigma = \sigma_1$, where $\sigma_1 > \sigma_0$, has the form

$$\frac{L(\sigma_0|\mathbf{x})}{L(\sigma_1|\mathbf{x})} < c \iff \frac{e^{-\frac{1}{2\sigma_0^2} \sum_{i=1}^n x_i^2}}{e^{-\frac{1}{2\sigma_1^2} \sum_{i=1}^n x_i^2}} < c_1 \iff \sum_{i=1}^n x_i^2 \left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2} \right) < c_2.$$

Finally, since $\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2} < 0$, we obtain that the critical region has the form $\sum_{i=1}^n x_i^2 > c_3$. The test is UMP, because the form of the most powerful critical region depends only on the fact that $\sigma_1 > \sigma_0$, but not on the specific value of σ_1 .