STAT2912: Statistical Tests

Semester 2	Statistical Tests (Advanced)	2011

Exercise 5 Solutions

1. From the lemma we have that

$$W^+ = \sum_{i=1}^{n}\sum_{j=1}^{i} I_{\{Z_i+Z_j>0\}} = \sum_{i=1}^{n}\sum_{j=1}^{i-1} I_{\{Z_i+Z_j>0\}} + \sum_{i=1}^{n} I_{\{Z_i>0\}}.$$

Therefore,

$$\begin{aligned}
E(W^+) &= \sum_{i=1}^{n}\sum_{j=1}^{i-1} P(Z_i+Z_j>0) + \sum_{i=1}^{n} P(Z_i>0) \\
&= P(Z_1+Z_2>0)\sum_{i=1}^{n}(i-1) + nP(Z_1>0) \\
&= \frac{n(n-1)}{2}P(Z_1+Z_2>0) + \frac{n}{2} \\
&= \frac{n(n-1)}{2}\frac{1}{2} + \frac{n}{2} = \frac{n(n+1)}{4},
\end{aligned}$$

where the first equation follows from the linearity of the expected value, the second from the fact that $Z_1, \ldots, Z_n$ are identically distributed, the third from the null hypothesis that Z_i's have median 0, and the fourht from the fact that Z_i's are assumed symmetric (about 0) r.v.'s.

2. This is done in the lecture notes. Here are some additional details:

$$\begin{aligned}
r_1 &= \frac{1+2+\cdots+12}{12} = 6.5 \\
r_2 &= \frac{13+14+\cdots+28}{16} = 20.5 \\
r_3 &= \frac{29+30+\cdots+59}{31} = 44 \\
r_4 &= \frac{60+61+\cdots+80}{21} = 70 \\
W_X &= 5r_1 + 7r_2 + 16r_3 + 12r_4 = 1720 \\
E(W_X) &= \frac{40(40+40+1)}{2} = 1620 \\
V(W_X) &= \frac{40(40)}{(40+40)(40+40+1)}\left[12r_1^2 + 16r_2^2 + 31r_3^2 + 21r_4^2 - \frac{(40+40)(40+40+1)^2}{4}\right] \\
&= 9854.937
\end{aligned}$$

3. (a) Here $n = 12$ so $M = 78$. The ranked differences are

$$3.5, 4.0, 4.5, 4.6, 5.0, 5.9, 6.0, 6.1, 6.5, 7.0, 8.5, 10.5$$

The values of $2 \times A_{(i)}$ can be obtained by evaluating the different $(x_i + x_j),\ \ j \leq i$.

	3.5	4.0	4.5	4.6	5.0	5.9	6.0	6.1	6.5	7.0	8.5	10.5
3.5	7											
4.0	7.5	8										
4.5	8	8.5	9									
4.6	8.1	8.6	9.1	9.2								
5.0	8.5	9	9.5	9.6	10							
5.9	9.4	9.9	10.4	10.5	10.9	11.8						
6.0	9.5	10	10.5	10.6	11	11.9	12					
6.1	9.6	10.1	10.6	10.7	11.1	12	12.1	12.2				
6.5	10	10.5	11	11.1	11.5	12.4	12.5	12.6	13			
7.0	10.5	11	11.5	11.6	12	12.9	13	13.1	13.5	14		
8.5	12	12.5	13	13.1	13.5	14.4	14.5	14.6	15	15.5	17	
10.5	14	14.5	15	15.1	15.5	16.4	16.5	16.6	17	17.5	19	21

The required confidence interval is

$$[A_{(15)}, A_{(64)}] = [9.5/2, 14.5/2] = [4.75, 7.25].$$

The Hodges-Lehmann estimator for the average gain is $(A_{(39)} + A_{(40)}) = \frac{5.8+5.9}{2} = 5.85$.

(b) If $n = 12$ then $E(W^+) = 12 \times 13/4 = 39$ and $Var(W^+) = 12 \times 13 \times 25/24 = 162.5$. Thus using the normal approximation to the distribution of W^+

$$k \simeq 39 - 1.96\sqrt{162.5} = 14.014$$

and the upper bound is 63.985. Thus we have $\mathrm{P}(W^+ \leq 14.01) \approx 0.025$ and we could use $k = 14$ and $[A_{(14)}, A_{(64)}]$, which results in the same CI as in part (a).