Tutorial 9 Solutions

① a) Notice that

$$X^T X = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{pmatrix}$$

$$\implies (X^T X)^{-1} = \begin{pmatrix} n & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{pmatrix}^{-1} = \frac{1}{\det(X^T X)} \begin{pmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{pmatrix}$$

$$= \frac{1}{n\sum x_i^2 - \left(\sum x_i\right)^2} \begin{pmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{pmatrix} = \frac{1}{n S_{xx}} \begin{pmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{pmatrix}.$$

Also, $$X^T Y = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\implies (X^T X)^{-1} X^T Y = \frac{1}{n S_{xx}} \begin{pmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{pmatrix} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$= \frac{1}{n S_{xx}} \begin{pmatrix} n\bar{y}\sum x_i^2 - n\bar{x}\sum x_i y_i \\ n\sum x_i y_i - n\bar{x}.n\bar{y} \end{pmatrix} = \begin{pmatrix} \dfrac{\bar{y}\sum x_i^2 - n\bar{y}\bar{x}^2 - \bar{x}\left(\sum x_i y_i - n\bar{y}\bar{x}\right)}{S_{xx}} \\ \dfrac{\sum x_i y_i - n\bar{x}\bar{y}}{S_{xx}} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{y} - \bar{x}\dfrac{S_{xy}}{S_{xx}} \\ \dfrac{S_{xy}}{S_{xx}} \end{pmatrix} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix}.$$

b) This is a linear model between y and x^2

$$\implies \hat{\alpha} = \bar{y} - \frac{1}{n}\sum_{i=1}^n x_i^2 \cdot \frac{\sum_{i=1}^n y_i x_i^2 - n\bar{y}\cdot\frac{1}{n}\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^4 - n\left(\frac{\sum_{i=1}^n x_i^2}{n}\right)^2}$$

and $$\hat{\beta} = \frac{\sum_{i=1}^n y_i x_i^2 - n\bar{y}\frac{1}{n}\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^4 - n\left(\frac{\sum_{i=1}^n x_i^2}{n}\right)^2}$$

(2) a) Summary statistics are

$$\bar{x} = 6, \quad \bar{y} = 984.45, \quad S_{xx} = 110, \quad S_{xy} = 3662, \quad S_{yy} = 143204.7$$

$$\Rightarrow \quad \hat{\beta} = \frac{3662}{110} = \boxed{33.29}$$

and $\hat{\alpha} = 984.45 - 33.29(6) = \boxed{784.71}$

The fitted line is $\boxed{\hat{y} = 784.71 + 33.29x}$.

b) $\hat{\beta} = 33.29$

c) $$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = S_{yy} - \hat{\beta} S_{xy} = 143204.7 - 33.29(3662)$$
$$= 21296.72$$

$$\Rightarrow \quad \hat{\sigma}^2_{SLR} = \frac{\sum_i (y_i - \hat{y}_i)^2}{n-2} = \boxed{2366.3}, \quad \hat{\sigma}^2 = \frac{S_{yy}}{n-1} = \boxed{14320.47}$$

d) 95% CI for β is

$$\hat{\beta} \pm t_{n-2,\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{S_{xx}}}$$

From table, $t_{9,0.0025} = 2.262$

$$\Rightarrow \text{CI is } 33.29 \pm 2.262 \frac{\sqrt{2366.3}}{\sqrt{110}} = 33.29 \pm 10.5$$

$$= \boxed{(22.79, 43.79)}.$$

e) $\hat{y}_{1972} = 784.71 + 33.29(12) = \boxed{1184.19}$.

Notice the real number is much smaller due to the seat belt law enforcement.