The University of Sydney

MATH1001 Differential Calculus

(http://www.maths.usyd.edu.au/u/UG/JM/MATH1001/)

Semester1, 2012 Lecturer: H. Dullin, R. Marangell, D. Warren, Z. Zhang

Week 4 Tutorial

Assumed knowledge

Familiarity with $\sqrt{x}$, $\sin x$, $\cos x$, e^x and $\ln x$ (and their graphs) for relevant sets of real numbers x.

Objectives

By the end of Week 4, you should

(a) know the definition of $e^{i\theta}$ as $\cos\theta + i\sin\theta$ and be able to write a non-zero complex number in polar exponential form, $z = re^{i\theta}$;

(b) be able to use the expressions for $\cos\theta$ and $\sin\theta$ in terms of $e^{i\theta}$ to find formulas for powers of $\cos\theta$ and $\sin\theta$, and to find formulas for $\cos n\theta$ and $\sin n\theta$;

(c) know the definition of e^z for $z \in \mathbb{C}$ and be able to solve the equation $e^z = \alpha$ given $\alpha \in \mathbb{C}$;

(d) know that a function from a set A to a set B is a rule that assigns to *each* element of A *exactly one* element of B;

(e) know that the notation $f\colon A \to B$ indicates that f is a function from A to B, that A is called the domain of f, and that by definition the range of f is $\{\, f(x) \mid x \in A \,\}$, the subset of B consisting of all values the function can take;

(f) be able, given a formula defining a function of a real variable or a function of a complex variable, to identify the natural domain of the function, and find the range of the function (in simple cases);

(g) be able to represent a function $f\colon A \to B$, where A and B are sets of real numbers, by its graph $\{\, (x, f(x)) \mid x \in A \,\}$ in the xy-plane;

(h) be able to use the vertical line test to recognise when a curve in the plane represents the graph of a function of one real variable;

(i) know that the composite $g\circ f$ of the function $f\colon A \to B$ and the function $g\colon B \to C$ is the function from A to C given by the rule that $(g\circ f)(x) = g(f(x))$ for all $x \in A$, and be able to determine the range of $g\circ f$ in simple cases.

Preparatory questions to do before the tutorial

1. Write the following numbers in the form $re^{i\theta}$ where $r, \theta \in \mathbb{R}$ and $r > 0$. (That is, write them in polar exponential form.) Use the principal argument in each case.

(i) $\sqrt{3} - i$ *(ii)* $1 - i$ *(iii)* 2π

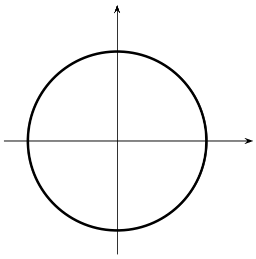
(iv) $(1 - i)(\sqrt{3} - i)$ *(v)* $(1 - i)/(\sqrt{3} - i)$ *(vi)* $(\sqrt{3} - i)^{19}$

2. In each part you are given the formula for calculating $f(x)$, where x is a real variable and f is to be a real-valued function. Find the natural domain of f.

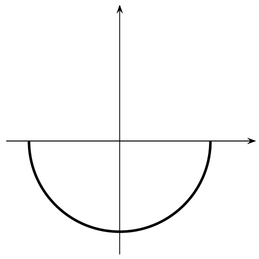
(i) $f(x) = \sqrt{x+1}$ *(ii)* $f(x) = \sqrt{4 - x^2}$ *(iii)* $f(x) = \sin|x|$ *(iv)* $f(x) = \ln(x + 1)$.

3. Which of the following graphs represent functions $f : A \to \mathbb{R}$ for some $A \subseteq \mathbb{R}$?

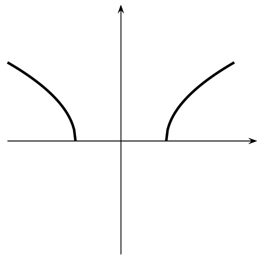
(i)



(ii)



(iii)



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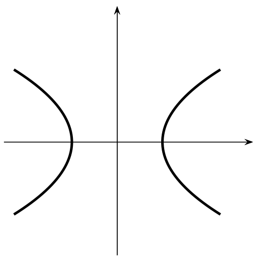

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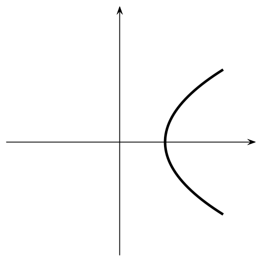
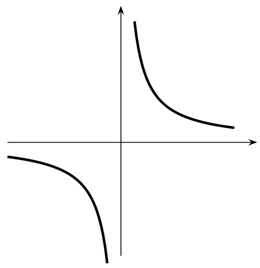


(vi)



Questions to do in the tutorial class

4. In each case find the natural domain of the function of one real variable determined by the given formula. Sketch the graph, and use it to find the range of the function.

(i) $f(x) = 1 + \cos x$ (ii) $f(x) = 3 - 2x$

(iii) $f(x) = \sqrt{x-5}$ (iv) $f(x) = \ln(1 - x^2)$

5. In each case find the natural domain of the function of a complex variable determined by the given formula. Find also the corresponding range. (For (iii), recall that if $z = x + iy$ in cartesian form, then $e^z = e^x(\cos y + i \sin y)$.)

(i) $f(z) = z^5$ (ii) $f(z) = \frac{1}{z}$ (iii) $f(z) = |e^z|$ (iv) $f(z) = e^{|z|}$.

6. Find formulas for $\sin 5\theta$ and $\cos 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$. (Hint: apply both de Moivre's theorem and the binomial theorem to $(\cos\theta + i\sin\theta)^5$, using the fact that the binomial coefficients $\binom{n}{k}$ for $n = 5$ are 1, 5, 10, 10, 5, 1.)

7. Find all solutions of

(i) $e^z = \sqrt{3} - i$ (ii) $e^z = -1$ (iii) $z^2 = e^{i\pi/4}$.

8. Using the formula $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, find an expression for $\sin^5\theta$ in terms of $\sin 5\theta$, $\sin 3\theta$ and $\sin\theta$. Use your answer to evaluate $\int_0^\pi \sin^5\theta \, d\theta$.

Questions for further practice

9. Find e^z for the following values of z.

(i) $z = 2i$ (ii) $z = 4 + \pi i$ (iii) $z = \cos\theta + i\sin\theta$ for real θ.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^2 + 3x - 4$. Find

(i) $f(0)$ (ii) $f(2)$ (iii) $f(\sqrt{2})$ (iv) $f(1 + \sqrt{2})$

(v) $f(-x)$ (vi) $f(x+1)$ (vii) $2f(x)$ $(viii)$ $f(2x)$.

11. Let $f(x) = \sin x$ and $g(x) = 1 - \sqrt{x}$, where x is a real variable. Write down the formulas for $f \circ g$ and $g \circ f$, and find the largest domain for which each composite function makes sense.

12. Find the ranges of the following functions. In each case take the domain of the function to be as large as possible.

(*i*) $f(x) = \sin|x|$ (*ii*) $g(x) = e^{\sqrt{x+1}}$.

13. Determine the natural domain and corresponding range for the three functions below. Assume that x is a real variable and z is a complex variable.

(*i*) $f(x) = (2+5x)^4$ (*ii*) $f(x) = \sqrt{e^x + 1}$ (*iii*) $f(z) = \frac{1}{e^z}$.

14. Let $S = \mathbb{R} \setminus \{0, 1\}$ and define functions $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ by the formulas $f(x) = \frac{1}{1-x}$ and $g(x) = \frac{x-1}{x}$ for all $x \in S$. (Note that S is not the natural domain in either case.)

(*i*) Find the ranges of f and g.

(*ii*) Find simplified rules for $f \circ g$ and $g \circ f$. What do you notice about the relationship between f and g?

Short answers to selected exercises

Full solutions can be downloaded at the end of the week.

1. (*i*) $2e^{-i\frac{\pi}{6}}$ (*ii*) $\sqrt{2}e^{-i\frac{\pi}{4}}$ (*iii*) $2\pi e^{i0}$
(*iv*) $2\sqrt{2}e^{-i\frac{5\pi}{12}}$ (*v*) $\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{12}}$ (*vi*) $2^{19}e^{i\frac{5\pi}{6}}$

2. (*i*) $[-1, \infty)$ (*ii*) $[-2, 2]$ (*iii*) $\mathbb{R}$ (*iv*) $(-1, \infty)$

3. Cases (*ii*), (*iii*) and (*vi*) are graphs of functions.

4. (*i*) Domain is $\mathbb{R}$, range is $[0, 2]$. (*ii*) Domain and range both $\mathbb{R}$.
(*iii*) Domain is $[5, \infty)$, range is $[0, \infty)$. (*iv*) Domain is $(-1, 1)$, range is $(-\infty, 0]$.

5. (*i*) Domain is $\mathbb{C}$, range is $\mathbb{C}$. (*ii*) Domain is $\mathbb{C} \setminus \{0\}$, range is $\mathbb{C} \setminus \{0\}$.
(*iii*) Domain is $\mathbb{C}$, range is $(0, \infty)$. (*iv*) Domain is $\mathbb{C}$, range is $[1, \infty)$.

6.
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
and
$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$$

7. (*i*) $z = \ln 2 + i(-\frac{\pi}{6} + 2k\pi)$; one solution for each $k \in \mathbb{Z}$.
(*ii*) $z = i\pi(2k+1)$; one solution for each $k \in \mathbb{Z}$.
(*iii*) Two solutions: $z = e^{i(\pi/8)}$ and $z = e^{i(-7\pi/8)}$.

8. $\sin^5\theta = \frac{1}{32}(2\sin 5\theta - 10\sin 3\theta + 20\sin\theta)$, and $\int_0^\pi \sin^5\theta\, d\theta = 16/15$.

9. (*i*) $-0.41 + 0.91i$ (approx.) (*ii*) $-e^4$
(*iii*) $e^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta))$

10. (*i*) -4 (*ii*) 10 (*iii*) $3\sqrt{2}$ (*iv*) $5 + 7\sqrt{2}$
(*v*) $2x^2 - 3x - 4$ (*vi*) $2x^2 + 7x + 1$ (*vii*) $4x^2 + 6x - 8$ (*viii*) $8x^2 + 6x - 4$

11. $(f \circ g)(x) = \sin\big(1 - \sqrt{x}\big)$; the domain is $[0, \infty)$. $(g \circ f)(x) = 1 - \sqrt{\sin x}$; the domain is the union of all the intervals of the form $[2k\pi, (2k+1)\pi]$ for $k \in \mathbb{Z}$.

12. (*i*) $[-1, 1]$. (*ii*) $[1, \infty)$.

13. (*i*) The domain is $\mathbb{R}$, the range is $[0, \infty)$.
(*ii*) The domain is $\mathbb{R}$, the range is $(1, \infty)$.
(*iii*) The domain is $\mathbb{C}$, the range is $\mathbb{C} \setminus \{0\}$.