THE UNIVERSITY OF SYDNEY

MATH1003 INTEGRAL CALCULUS AND MODELLING

Practice Questions for Quiz 2 2009

Quiz 2 will be held during your tutorial in week 11 (beginning 12 October). The quiz questions will be based on material covered during lectures in weeks 5–9, which corresponds to material covered in tutorials in weeks 6–10. They will also draw on knowledge of integration techniques from earlier weeks.

The quiz will run for 40 minutes. You may use a non-programmable calculator. No other materials are permitted.

1. Find dy/dx when

(a) $y = x^{1/x}$, (b) $y = \log_{10}(4x^3)$ $\left(= \dfrac{\ln(4x^3)}{\ln 10}\right)$,

(c) $y = \dfrac{x^3(10x-9)^{1/5}}{\sqrt{1+x^2+x^4}}$.

2. Use the definitions of the hyperbolic functions $\cosh x = \big(e^x + e^{-x}\big)/2$ and $\sinh x = \big(e^x - e^{-x}\big)/2$ to simplify the following expressions:

(a) $\cosh A \cosh B - \sinh A \sinh B$, (b) $\cosh^2\theta - \sinh^2\theta$,

(c) $\cosh^2\theta + \sinh^2\theta$.

Use the results of parts (b) and (c) to simplify

(d) $\cosh^4\theta - \sinh^4\theta$.

3. Use the substitution $x = \sin\theta$ to evaluate $\displaystyle\int \frac{x^2}{(1-x^2)^{5/2}}\,dx$.

4. Decompose the following expressions into partial fractions and then integrate them:

(a) $\dfrac{5x}{x^2+x-6}$, (b) $\dfrac{x^2}{(x-1)(x-2)}$.

5. Find the general solution of the equation

$$\frac{dy}{dx} = \frac{5x}{x^2+x-6}\,.$$

Find the particular solution satisfying the condition $y(3) = 1$.

6. Which of the following differential equations are separable? In the cases that are separable, write the differential equations in such a way that the variables are fully separated:

(a) $\dfrac{dy}{dx} = 1 + 2x + y^2 + 2xy^2$, (b) $\dfrac{dy}{dx} = x^2 + y^2$,

(c) $e^{-t}\dfrac{dw}{dt} = 1 + w^2 \sin t$, (d) $\dfrac{dv}{du} = \cos(u+v) + \cos(u-v)$.

7. Solve the separable differential equations in the previous question.

8. Which of the following differential equations are linear? In the cases that are linear, write the differential equations in standard form:

(a) $\dfrac{dy}{dx} = x^2 + y^2$, (b) $\dfrac{dy}{dx} = x + y$,

(c) $e^x\left(\dfrac{dy}{dx} + 2y\right) = 1$, (d) $\dfrac{dy}{dx} = x^3y^3$.

9. Solve the linear differential equations (in standard form) in the previous question. In each case, give the integrating factor and find y explicitly as a function of x.

10. An isolated colony of animals has a birth rate and a death rate that are proportional to the total population. The population doubles every 30 years. At time $t = 0$, the population is 1000 animals. Calculate the population after 40 years and the time taken for the population to increase tenfold.

ANSWERS

1. (Use logarithmic differentiation for parts (a) and (c)).

(a) $x^{1/x}\dfrac{1-\ln x}{x^2}$, (b) $\dfrac{3}{x\ln 10}$, (c) $\dfrac{x^3(10x-9)^{1/5}}{\sqrt{1+x^2+x^4}}\left\{\dfrac{3}{x} + \dfrac{2}{10x-9} - \dfrac{x(1+2x^2)}{1+x^2+x^4}\right\}$.

2. (a) $\cosh(A-B)$, (b) 1, (c) $\cosh 2\theta$, (d) $\cosh 2\theta$.

3. $$\int \frac{x^2}{(1-x^2)^{5/2}}\,dx = \int \frac{\sin^2\theta}{\cos^4\theta}\,d\theta = \int \tan^2\theta\sec^2\theta\,d\theta = \frac{\tan^3\theta}{3} + C$$
$$= \frac{x^3}{3(1-x^2)^{3/2}} + C.$$

4. (a) $\dfrac{2}{x-2} + \dfrac{3}{x+3}$, integral: $2\ln|x-2| + 3\ln|x+3| + C$,

(b) $1 - \dfrac{1}{x-1} + \dfrac{4}{x-2}$, integral: $x - \ln|x-1| + 4\ln|x-2| + C$.

5. General solution: $y = 2\ln|x-2| + 3\ln|x+3| + C$ (see Q4(a)). Particular solution: $y = 2\ln|x-2| + 3\ln|x+3| + 1 - 3\ln 6$ which simplifies to

$y = 2\ln(x-2) + 3\ln(x+3) + 1 - 3\ln 6$ as the initial condition $y(3) = 1$ implies that $x \geq 3$.

6. (a) Separable: $\dfrac{dy}{1+y^2} = (1+2x)\,dx$, (b) not separable,

(c) not separable, (d) separable: $\dfrac{dv}{\cos v} = 2\cos u\,du$.

7. (a) $y = \tan(x^2 + x + C)$, (d) $\sec v + \tan v = C\,e^{2\sin u}$.

Comment. Although this would not be required in the quiz, there are several ways to find an explicit formula for v in part (d). We have $(1+\sin v)/\cos v = C\,e^{2\sin u}$. Then, for example, following Q4 in tutorial week 9, if $t = \tan(v/2)$, $\tan(v/2) = (1-\cos v)/\sin v$ using the formulae $\cos v = (1-t^2)/(1+t^2)$ and $\sin v = 2t/(1+t^2)$. It follows that $\tan((v+\pi/2)/2) = (1+\sin v)/\cos v$ so that $\tan((v+\pi/2)/2) = C\,e^{2\sin u}$. Finally $v = 2\tan^{-1}\big(C\,e^{2\sin u}\big) - \frac{1}{2}\pi$. Other methods lead to different, but equivalent forms.

8. (a) Nonlinear, (b) linear, standard form: $\dfrac{dy}{dx} - y = x$,

(c) linear, standard form: $\dfrac{dy}{dx} + 2y = e^{-x}$, (d) nonlinear.

9. (b) Integrating factor e^{-x}, solution $y = -(x+1) + Ce^{x}$,
(c) integrating factor e^{2x}, solution $y = e^{-x} + Ce^{-2x}$.

10. Let the population be $P(t)$. The differential equation is $dP/dt = kP$, with solution $P(t) = 1000\,e^{kt}$. Population doubling every 30 years implies $e^{30k} = 2$, and so $P(t) = 1000 \cdot 2^{t/30}$. After 40 years, $P(40) = 1000 \cdot 2^{4/3} = 2520$. The population reaches 10000 when $t = 30\ln 10/\ln 2 = 99.66$ years.