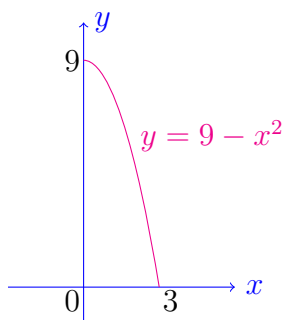


1. (i) [**3 Marks**] Sketch the curve  $y = 9 - x^2$  over the interval  $[0, 3]$ . Give the expression of the area under the curve in terms of a definite integral. Then use the Fundamental Theorem of Calculus to calculate the area under the curve.
- (ii) [**2 Marks**] Evaluate the lower Riemann sum for  $f(x) = 9 - x^2$  on  $[0, 3]$  by dividing this interval into 4 subintervals of equal length.
- (iii) [**2 Marks**] Divide the interval  $[0, 3]$  into  $n$  subintervals of equal length. Prove that for this partition of  $[0, 3]$  the lower Riemann sum  $L_n$  for  $f(x) = 9 - x^2$  equals

$$L_n = \frac{3}{n} \sum_{k=1}^n \left[ 9 - \left( \frac{3k}{n} \right)^2 \right].$$

- (iv) [**1 Mark**] Without evaluating the limit directly, explain why as  $n$  tends to  $\infty$ , the limit of the lower Riemann sum  $L_n$  equals 18. (Hint: Use a theorem from lectures).

**Solution.** (i) Notice that  $f(x) = 9 - x^2$  is a continuous function, which is decreasing on  $[0, 3]$ . Since  $f(3) = 0$ , the graph of  $f$  on  $[0, 3]$  lies above the  $x$ -axis.



The required area is given by the definite integral of  $f$  over the interval  $[0, 3]$ . This is calculated using the Fundamental Theorem of Calculus, namely

$$\int_0^3 (9 - x^2) dx = \left( 9x - \frac{x^3}{3} \right) \Big|_0^3 = 18.$$

- (ii) The partition of the interval  $[0, 3]$  into 4 equal subintervals is given by

$$x_0 = 0 < x_1 = 0.75 < x_2 = 1.5 < x_3 = 2.25 < x_4 = 3.$$

Notice that  $\Delta x = x_k - x_{k-1} = 0.75$  for each  $k$  from 1 to 4. The lower Riemann sum for  $f$  corresponding to the above partition is thus given by

$$L_4 = \sum_{k=1}^4 0.75 \min_{x \in [x_{k-1}, x_k]} f(x).$$

Since the function  $f$  is decreasing on  $[0, 3]$ , the minimum of  $f(x)$  on  $[x_{k-1}, x_k]$  is achieved at  $x = x_k$  (the right endpoint of the interval). It follows that  $L_4$  equals to

$$0.75 [f(0.75) + f(1.5) + f(2.25) + f(3)] = 0.75(8.4375 + 6.75 + 3.9375 + 0) = 14.34375$$

(iii) If we divide the interval  $[0, 3]$  into  $n$  subintervals of equal length, then each subinterval will have the length  $\Delta x = \frac{3}{n}$  (the difference between the endpoints of the interval divided by the number of subintervals). The partition of the interval  $[0, 3]$  is

$$0 = x_0 < x_1 < x_2 < \dots < x_{k-1} < x_k < \dots < x_n = 3,$$

where  $x_k = x_{k-1} + \Delta x$  for each  $1 \leq k \leq n$ . So, for each integer  $k$  from 1 to  $n$ , we have

$$x_k = x_0 + k\Delta x = \frac{3k}{n}. \quad (1)$$

Since the function  $f$  is decreasing on  $[0, 3]$ , the minimum of  $f$  on each subinterval  $[x_{k-1}, x_k]$  is given by the value of  $f$  at the right endpoint of the interval, that is  $f(x_k)$ . Using also (1), the lower Riemann sum for  $f$  on  $[0, 3]$  corresponding to our partition is

$$L_n = \sum_{k=1}^n \min_{x \in [x_{k-1}, x_k]} f(x) \Delta x = \sum_{k=1}^n f(x_k) \frac{3}{n} = \frac{3}{n} \sum_{k=1}^n f\left(\frac{3k}{n}\right).$$

(iv) The limit of the lower Riemann  $\lim_{n \rightarrow \infty} L_n$  equals the definite integral of  $f$  over  $[0, 3]$ , which we know to be equal to 18 from Part (i).

**2.** Let  $f$  be a continuous function on  $[-a, a]$  which is an odd function, that is

$$f(-x) = -f(x) \quad \text{for all } x \text{ in the interval } [-a, a].$$

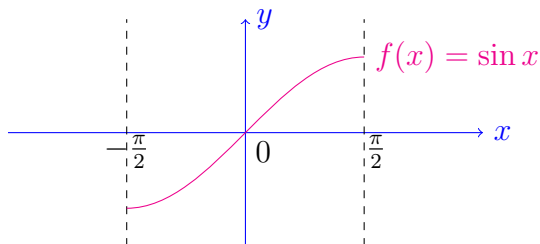
- (a) **[1 Mark]** Give a geometric interpretation of why  $\int_{-a}^a f(x) dx = 0$ . (Give a sketch).  
 (b) **[2 Marks]** By using a suitable substitution, prove that

$$\int_{-a}^0 f(x) dx = - \int_0^a f(x) dx,$$

whenever  $f$  is a continuous and odd function on  $[-a, a]$ .

- (c) **[1 Mark]** Evaluate the integral  $\int_{-\pi/2}^{\pi/2} (\sin x)^5 dx$ .

**Solution.** (a) We illustrate below the graph of the function  $\sin x$  on  $[-\pi/2, \pi/2]$ .



Notice that  $f(0) = 0$  and the graph of  $f$  is symmetric about the origin. If the graph of  $f$  over  $[0, a]$  is reflected into  $x$ -axis, then into  $y$ -axis we obtain the graph of  $f$  over  $[-a, 0]$ . We interpret  $\int_{-a}^a f(x) dx$  as the area under the graph of  $f$  over  $[-a, a]$ , where the area below the  $x$ -axis counts as negative. Hence, we find  $\int_{-a}^a f(x) dx = 0$  since

$$\int_{-a}^0 f(x) dx = - \int_0^a f(x) dx.$$

(b) We use the substitution  $t = -x$  in the first integral. Since  $dt = -dx$  and  $t = a$  when  $x = -a$ , it follows that

$$\int_{-a}^0 f(x) dx = - \int_a^0 f(-t) dt.$$

We use the sign minus in front of the above integral to switch the limits of integration, then using the property that  $f(-t) = -f(t)$ , we conclude that

$$\int_{-a}^0 f(x) dx = - \int_0^a f(t) dt.$$

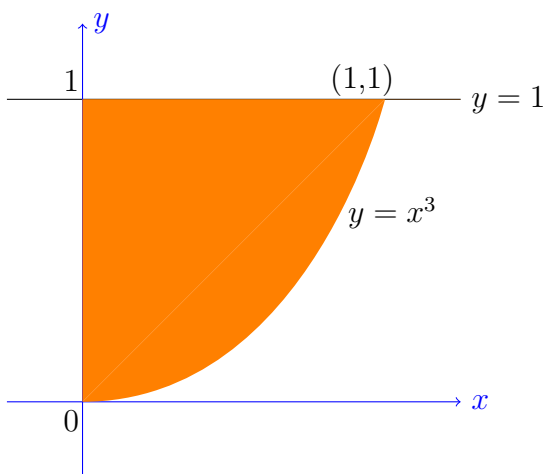
(c) Since  $\sin x$  is an odd function, we see that  $f(x) = (\sin x)^5$  is also an odd function:

$$f(-x) = (\sin(-x))^5 = (-\sin x)^5 = -(\sin x)^5.$$

Hence from Part (b), we conclude that  $\int_{-\pi/2}^{\pi/2} (\sin x)^5 dx = 0$ .

3. (i) **[1 Mark]** Sketch the area enclosed by the positive  $y$  axis, the line  $y = 1$  and the curve  $y = x^3$ .
- (ii) A solid of rotation  $S$  is formed by rotating this area about the  $y$ -axis.
- (a) **[1 Mark]** For each  $y$  in  $[0, 1]$  on the  $y$ -axis, the cross sectional area of  $S$  perpendicular to the  $y$ -axis is denoted by  $A(y)$ . Find a formula for  $A(y)$ .
- (b) **[2 Marks]** Derive an expression approximating the volume  $\Delta V$  of the thin solid obtained by rotating a thin horizontal strip of the area of width  $\Delta y$  located at  $y$ .
- (c) **[4 Marks]** Hence determine a definite integral which will give the total volume of the solid and evaluate this integral.

**Solution.** (i) Let  $R$  denote the region enclosed by the positive  $y$  axis, the line  $y = 1$  and curve  $y = x^3$ . This region is depicted below as the shaded area.



(ii) (a) When the region  $R$  is revolved about  $y$ -axis, it generates a solid  $S$ . At any point  $y$  in  $[0, 1]$  on the  $y$ -axis, the cross section of  $S$  perpendicular to the  $y$ -axis is a disk of radius  $y^{1/3}$ . Hence the required cross-sectional area  $A(y)$  is given by  $\pi y^{2/3}$ .

(b) By rotating about  $y$ -axis a thin horizontal strip of the area of width  $\Delta y$  located at  $y$ , we obtain a solid whose volume  $\Delta V$  is approximated by

$$\pi y^{2/3} \Delta y.$$

This value is the volume of a right circular cylinder obtained by rotating about  $y$ -axis a rectangle located at  $y$  of width  $\Delta y$  and height  $y^{1/3}$ .

(c) To find the volume of the solid of rotation, we divide the interval  $[0, 1]$  (on the  $y$ -axis) into  $n$  subintervals with widths

$$\Delta y_1, \Delta y_2, \dots, \Delta y_n$$

by inserting points  $y_1, y_2, \dots, y_{n-1}$  between  $0 = y_0$  and  $1 = y_n$ . We partition the interval  $[0, 1]$  in such a way that its norm (the length of the longest subinterval) tends to zero as  $n$  increases to  $\infty$ . Through each of the points of the partition we draw a line parallel to  $x$ -axis such that the region  $R$  is divided into  $n$  strips  $R_1, R_2, \dots, R_n$ . These strips when rotated about the  $y$ -axis will generate solids  $S_1, S_2, \dots, S_n$ . For each  $1 \leq k \leq n$ , we approximate the volume of  $S_k$  as in Part (b) by rotating a rectangle of width  $\Delta y_k$  and height  $y_k^{1/3}$ :

$$\text{Vol}(S_k) \sim \pi y_k^{2/3} \Delta y_k.$$

By adding up the volumes of all solids  $S_1, S_2, \dots, S_n$ , we get the volume of the entire solid.

$$\text{Vol}(S) = \sum_{k=1}^n \text{Vol}(S_k) \sim \sum_{k=1}^n \pi y_k^{2/3} \Delta y_k.$$

The approximation gets better and better as  $n \rightarrow \infty$  so that  $\max_{1 \leq k \leq n} \Delta y_k \rightarrow 0$  and

$$\text{Vol}(S) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \pi y_k^{2/3} \Delta y_k.$$

We thus obtain the volume of rotation as the following definite integral

$$\int_0^1 \pi y^{2/3} dy = \frac{3\pi}{5} y^{5/3} \Big|_0^1 = \frac{3\pi}{5}.$$