THE UNIVERSITY OF SYDNEY

MATH1003 Integral Calculus and Modelling

Semester 2 **Solutions to Assignment 2** 2009

---

1. (a) [3 marks] Find the general solution to $(x^2+x-2)\dfrac{dy}{dx} = xy^2$.

(b) [3 marks] Find the particular solution to $\dfrac{dy}{dx} - y\tan x = x$, given that $y(0) = 0$.

Solution

(a) The differential equation can be written in the form,

$$\frac{dy}{dx} = \frac{x}{x^2+x-2}\,y^2,$$

where the right-hand side is a function of x times a function of y. So the equation is separable. Separating variables and integrating gives

$$\begin{aligned}\frac{dy}{y^2} &= \frac{x\,dx}{x^2+x-2}\\ &= \frac{x\,dx}{(x-1)(x+2)},\\ \int\frac{dy}{y^2} &= \int\frac{x\,dx}{(x-1)(x+2)}.\end{aligned}$$

Rewrite the integrand on the right-hand side in terms of its partial fraction decomposition. We get

$$\begin{aligned}\frac{x}{(x-1)(x+2)} &= \frac{A}{x-1}+\frac{B}{x+2}\\ &= \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}\\ &= \frac{(A+B)x+2A-B}{(x-1)(x+2)}.\end{aligned}$$

The two sides agree if A and B satisfy

$$A+B=1, \qquad 2A-B=0.$$

Solving for A and B gives

$$A = \frac{1}{3}, \qquad B = \frac{2}{3}.$$

Before continuing with the solution of the differential equation, let us obtain A and B by an alternative method, which would be considerably quicker in cases where the polynomial on the denominator is a cubic or higher-degree polynomial. We wish to find A and B such that

$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{x}{(x-1)(x+2)}.$$

Multiply both sides by $x-1$ and put $x=1$. The result is

$$A = \left[\frac{x}{x+2}\right]_{x=1} = \frac{1}{3}.$$

Similarly,

$$B = \left[\frac{x}{x-1}\right]_{x=-2} = \frac{2}{3}.$$

Either way, we have

$$\frac{x}{(x-1)(x+2)} = \frac{1}{3}\left(\frac{1}{x-1} + \frac{2}{x+2}\right).$$

Returning to the differential equation, we get the general solution,

$$\int \frac{dy}{y^2} = \int \frac{1}{3}\left(\frac{1}{x-1} + \frac{2}{x+2}\right) dx,$$

$$-\frac{1}{y} = \frac{1}{3}\big\{\ln|x-1| + 2\ln|x+2|\big\} + C,$$

$$y = -\frac{3}{\ln|x-1| + 2\ln|x+2| + 3C}.$$

You can stop here. On the other hand, if you think "$3C$" looks a little inelegant, you can rename the constant, say, $K = 3C$. Then the final form of the general solution is

$$y = -\frac{3}{\ln|x-1| + 2\ln|x+2| + K}.$$

(b) The given differential equation,

$$\frac{dy}{dx} - y\tan x = x,$$

is a first-order linear differential equation, and is already in standard form. Its integrating factor is

$$r(x) = \exp\left(\int -\tan x \, dx\right) = \exp\left(\ln \cos x\right) = \cos x.$$

Multiply both sides by the integrating factor:

$$\begin{aligned} \cos x \frac{dy}{dx} - (\sin x)y &= x \cos x, \\ \frac{d}{dx}\left(y \cos x\right) &= x \cos x. \end{aligned}$$

Integrate both sides with respect to x and use integration by parts on the right-hand side:

$$\begin{aligned} y \cos x &= \int x \cos x \, dx \\ &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C, \\ y &= x \tan x + 1 + C \sec x. \end{aligned}$$

This is the general solution. We want the particular solution such that $y = 0$ when $x = 0$, which forces $C = -1$. Therefore the required particular solution is

$$y = x \tan x + 1 - \sec x.$$

2. A simple physiological model suggests that an adult male athlete needs 40 calories per day per kilogram of body weight in order to maintain his weight. If he consumes more, or fewer, than this amount of calories, then his weight changes at a rate which is proportional to the difference between the number of calories consumed, and the number required to maintain his weight. The constant of proportionality is $\frac{1}{7000}$ (kilograms per calorie). Let "P" denote a particular male athlete who has a constant intake of L calories per day.

(a) [2 marks] Assuming this model, write a differential equation that describes the rate of change of P's weight. (Don't forget to name your variables.)

(b) [3 marks] Solve your differential equation.

(c) [2 marks] If P currently weighs 80 kilograms, and consumes 3000 calories a day for the next 90 days, what will his weight be at the end of those 90 days?

Solution

(a) Let W be P's weight. Let t be the time in days. The model implies that dW/dt is proportional to $L - 40W$ with constant of proportionality $1/7000$. Hence, $W(t)$ satisfies the differential equation,

$$\frac{dW}{dt} = \frac{1}{7000}(L - 40W).$$

(b) This differential equation is both separable and linear. Let us treat it as linear. (You should also treat it as separable and compare the two solutions, which should agree except possibly for the name of the constant of integration.) The standard form is

$$\frac{dW}{dt} + \frac{1}{175}W = \frac{L}{7000}.$$

The integrating factor is $e^{t/175}$. Hence, the differential equation takes the form,

$$\frac{d}{dt}\left\{e^{t/175}W\right\} = \frac{L}{7000}\,e^{t/175}.$$

Integrating both sides gives the general solution:

$$\begin{aligned} e^{t/175}W &= \frac{L}{40}\,e^{t/175} + C, \\ W(t) &= \frac{L}{40} + Ce^{-t/175}. \end{aligned}$$

(c) Introduce the given initial condition $W = 80$ when $t = 0$. Substituting into the general solution identifies $C = 80 - L/40$. So far, the particular solution reads

$$W(t) = \frac{L}{40} + \left(80 - \frac{L}{40}\right)e^{-t/175}.$$

We are also given the specific value $L = 3000$. The required solution becomes

$$W(t) = 75 + 5e^{-t/175}.$$

We want to know the value of P's weight after 90 days. Putting $t = 90$ and using a calculator gives

$$W(90) = 75 + 5e^{-18/35} \approx 77.99.$$

After 90 days, P's weight will be about 78 kilograms. (So he has lost only 2 kilograms. This is to be expected since 3000 calories per day is only a little less than his equilibrium rate of 3200 calories per day.)

3. A tank with a capacity of 1000 litres contains 600 litres of a mixture of water and chlorine. The concentration of the chlorine is 0.05 grams per litre. In order to reduce the chlorine concentration, fresh water is pumped into the tank at a rate of 6 litres per second for 3 minutes. The mixture in the tank is kept well-stirred, and is pumped out at the rate of 4 litres per second during the same 3 minutes.

(a) [4 marks] Find the mass of chlorine in the tank as a function of time.

(b) [3 marks] What is the concentration of chlorine in the tank at the end of the 3 minutes?

Solution

(a) Let $m(t)$ be the mass of chlorine and $V(t)$ the volume of liquid in the tank at time t, where t is the time in seconds. Then

$$\frac{dm}{dt} = \text{rate that chlorine flows in} - \text{rate that chlorine flows out.}$$

Since only fresh water flows into the tank, the rate that chlorine flows into the tank is zero. The rate that chlorine flows out is given by the concentration of chlorine in the tank multiplied by the rate of outflow of liquid. That is,

$$\text{rate that chlorine flows out} = \frac{m}{V} \times 4 = \frac{4m}{V}.$$

But V itself is a function of t because

$$\begin{aligned}\frac{dV}{dt} &= \text{rate of volume flowing in} - \text{rate of volume flowing out} \\ &= 6 - 4 \\ &= 2.\end{aligned}$$

So the initial volume of 600 litres implies that $V(t) = 600 + 2t$.

So the rate that the chlorine flows out of the tank is

$$\frac{4m}{V} = \frac{4m}{2t + 600} = \frac{2m}{t + 300}.$$

The differential equation for $m(t)$ now takes the form,

$$\frac{dm}{dt} = -\frac{2m}{t + 300}.$$

This differential equation is both separable and linear. Let us treat it as separable:

$$\begin{aligned}
\frac{dm}{m} &= -\frac{2\,dt}{t+300}, \\
\int \frac{dm}{m} &= -\int \frac{2\,dt}{t+300}, \\
\ln m &= -2\ln(t+300) + C, \\
m(t) &= \frac{A}{(t+300)^2},
\end{aligned}$$

where $A = e^C$. (We disregarded absolute value signs on the quantities inside the logarithms because they are guaranteed to be positive.) At $t = 0$, we are given that the concentration of chlorine in the tank is 0.05 grams per litre. This means that the initial mass of chlorine (in grams) is $m(0) = 0.05 \times 600 = 30$. So the constant A is given by

$$A = 30 \times 300^2 = 2700000.$$

The mass of chlorine in the tank at time t is

$$m(t) = \frac{2700000}{(t+300)^2}.$$

(b) After three minutes (that is, at $t = 180$ seconds), the mass of chlorine in the tank is

$$m(180) = \frac{2700000}{480^2} = 11.71875.$$

So there are approximately 11.7 grams of chlorine in the tank after three minutes. The question asked for the concentration rather than the mass. To get the concentration, we need to divide the mass by the volume of liquid at that time. Recall that $V(t) = 600 + 2t$. At $t = 180$, we have $V = 960$, which incidentally is only 40 litres short of the maximum capacity of the tank. So the concentration of chlorine in the tank after three minutes is

$$\text{Concentration} = \frac{11.71875}{960} \approx 0.0122 \text{ grams per litre.}$$