THE UNIVERSITY OF SYDNEY
MATH1003, Integral Calculus and Modelling (Normal)
Solutions to Extended Answer Section 2008

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1. (a) [4 marks.] The partial fraction decomposition will take the form,

$$\frac{1}{x(4-x)} = \frac{A}{x} + \frac{B}{4-x}.$$

The quick way to find A and B is as follows:

$$A = \left[\frac{1}{4-x}\right]_{x=0} = \frac{1}{4},$$
$$B = \left[\frac{1}{x}\right]_{x=4} = \frac{1}{4}.$$

The standard method (which is quite efficient in this example) is

$$\begin{aligned} 1 &= A(4-x) + Bx \\ &= (B-A)x + 4A. \end{aligned}$$

Equating coefficients gives

$$A = \frac{1}{4}, \qquad B = A = \frac{1}{4}.$$

Hence, the required decomposition is

$$\boxed{\frac{1}{x(4-x)} = \frac{1}{4}\left(\frac{1}{x} + \frac{1}{4-x}\right).}$$

Remark. It is equally correct to write the decomposition in the form $A/x + B/(x-4)$, in which case $A = 1/4$ and $B = -1/4$. Then the decomposition would be written: $(1/4)\big(1/x - 1/(x-4)\big)$. But, looking ahead, we see that the domain of interest is $1 \leq x \leq 3$ and so the decomposition in the box is preferable because integration yields logarithms with positive arguments that do not need absolute value signs.

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The discontinuities occur at $x = 0$ and $x = 4$, and neither of these points occur on the closed interval $[1, 3]$. So the following Riemann integral exists:

$$\begin{aligned}
\int_1^3 \frac{dx}{x(4-x)} &= \frac{1}{4}\int_1^3 \left(\frac{1}{x} + \frac{1}{4-x}\right) dx \\
&= \frac{1}{4}\Big[\ln|x| - \ln|4-x|\Big]_1^3 \qquad \text{(absolute value signs optional)} \\
&= \frac{1}{4}\Big[\ln x - \ln(4-x)\Big]_1^3 \\
&= \frac{1}{4}\Big(\ln 3 - \ln 1 - (\ln 1 - \ln 3)\Big) \\
&= \frac{\ln 3}{2}.
\end{aligned}$$

If $\ln|x-4|$ is used instead of $\ln(4-x)$, then the absolute value signs are important, but the calculation is essentially the same. Either way, the answer is

$$\boxed{\int_1^3 \frac{dx}{x(4-x)} = \frac{\ln 3}{2}.}$$

(b) [3 marks.] Change sign and complete the square:

$$x(x-4) = x^2 - 4x = (x^2 - 4x + 4) - 4 = (x-2)^2 - 4.$$

Hence $a = b = 2$ and the required expression is

$$\boxed{x(4-x) = 2^2 - (x-2)^2.}$$

The expression $4 - (x-2)^2$ is also acceptable on the right-hand side. Using the standard integral tables, we can calculate the required integral on the domain $0 < x < 4$ as follows:

$$\begin{aligned}
\int \frac{dx}{\sqrt{x(4-x)}} &= \int \frac{dx}{\sqrt{2^2 - (x-2)^2}} \\
&= \int \frac{du}{\sqrt{2^2 - u^2}} \qquad \text{(substitute } x = u + 2,\ dx = du\text{)} \\
&= \sin^{-1}\left(\frac{u}{2}\right) + C \\
&= \sin^{-1}\left(\frac{x-2}{2}\right) + C.
\end{aligned}$$

The answer is

$$\boxed{\int \frac{dx}{\sqrt{x(4-x)}} = \sin^{-1}\left(\frac{x-2}{2}\right) + C.}$$

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(c) [5 marks.] For the differential equation,

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 0,$$

the auxiliary equation is

$$m^2 + 4m + 5 = 0.$$

Completing the square gives $m^2 + 4m + 5 = (m+2)^2 + 1$, and so the roots of the auxiliary equation are $m = -2 \pm i$. Without any further calculation, we can read off the general solution,

$$\boxed{x(t) = e^{-2t}\big(A\cos t + B\sin t\big).}$$

To satisfy the given initial conditions, we also need the first derivative,

$$\begin{aligned} x'(t) &= -2e^{-2t}\big(A\cos t + B\sin t\big) + e^{-2t}\big(-A\sin t + B\cos t\big) \\ &= e^{-2t}\big\{(B-2A)\cos t - (A+2B)\sin t\big\}. \end{aligned}$$

Substituting the initial conditions $x(0) = 3$ and $x'(0) = 6$, we get

$$A = 3, \qquad B - 2A = 6,$$

which give $A = 3$ and $B = 12$. It follows that the required particular solution is

$$\boxed{x(t) = 3e^{-2t}\big(\cos t + 4\sin t\big).}$$

2. (a) The region in the xy-plane defined by $-x^2 \le y \le x^2$ and $0 \le x \le L$ is to be rotated about the vertical line $x = L$ to form a solid of revolution.

(*i*). [4 marks.] The vertical strip has width Δx (presumed small) and height $2x^2$. Its distance from the given vertical line $x = L$ is $L - x$ (beware two different usages of the notation x here). The volume of the cylindrical shell formed when the strip is rotated about the line $x = L$ is

$$\begin{aligned} \Delta V &= 2\pi(\text{radius})(\text{height})(\text{thickness}) \\ &= 2\pi(L-x)(2x^2)\Delta x \\ &= \boxed{4\pi x^2(L-x)\Delta x.} \end{aligned}$$

(*ii*). [2 marks.] The result of part (*i*) can be used to form a Riemann sum for the function $f(x) = 4\pi x^2(L-x)$ on the interval $[0, L]$. The limit of the sum as the partitions are refined is the Riemann integral,

$$\begin{aligned} V &= 4\pi \int_0^L x^2(L-x)\,dx \\ &= 4\pi \left[\frac{1}{3}Lx^3 - \frac{1}{4}x^4\right]_0^L \\ &= 4\pi \left(\frac{1}{3}L^4 - \frac{1}{4}L^4\right) \\ &= \frac{\pi L^4}{3}. \end{aligned}$$

Thus the volume of the solid of revolution is

$$\boxed{V = \frac{\pi L^4}{3}.}$$

(b) (*i*). [2 marks.]

$$J = \int_0^{1000} x^{2/3}\,dx = \left[\frac{3}{5}x^{5/3}\right]_0^{1000} = \frac{3}{5}1000^{5/3} = \frac{3}{5}10^5 = \boxed{60000.}$$

(*ii*). [2 marks.] The function $f(x) = x^{2/3}$ is increasing on the interval $[0, 1000]$. Take the partition $\{0, 1, 2, 3, \ldots, 1000\}$ of the interval. The common width of each subinterval is $\Delta x = 1$. For the lower Riemann sum L_{1000}, the sample point x_j^* in the jth subinterval $[j-1, j]$ will be the left endpoint $j-1$. For the upper Riemann sum U_{1000}, we will use the right endpoint j. The lower Riemann sum is

$$\begin{aligned} L_{1000} &= \sum_{j=1}^{1000} f(j-1)\Delta x = \sum_{j=1}^{1000} (j-1)^{2/3} \\ &= 0 + 1 + 2^{2/3} + 3^{2/3} + \ldots + 999^{2/3} = S_{999}. \end{aligned}$$

The corresponding upper Riemann sum is

$$\begin{aligned} U_{1000} &= \sum_{j=1}^{1000} f(j)\Delta x = \sum_{j=1}^{1000} j^{2/3} \\ &= 1 + 2^{2/3} + 3^{2/3} + \ldots + 1000^{2/3} = S_{1000}. \end{aligned}$$

Since the exact integral J is between these extremes, we have the inequality,

$$\boxed{S_{999} < 60000 < S_{1000}.}$$

(iii). [2 marks.] Now $S_{1000} - S_{999} = 1000^{2/3} = 10^2 = 100$. Adding 100 to both sides of $S_{999} < 60000$ gives $S_{1000} < 60100$. Rearranging gives the bounds,

$$\boxed{60000 < S_{1000} < 60100,}$$

where the upper and lower bounds differ by 100, the maximum allowed. [*Remark.* Because $f(x) = x^{2/3}$ is concave down, it can be shown that the mid-value 60050 is also an upper bound for S_{1000}. It is also accurate to one part in 400000.]

3. (a) [4 marks.] The given differential equation is separable. Write it as

$$e^{-y}\,dy = \frac{dx}{x^2+1}.$$

Integrating both sides gives

$$\begin{aligned} -e^{-y} &= \tan^{-1} x + C, \\ e^{-y} &= -\big(\tan^{-1} x + C\big). \end{aligned}$$

The condition $y = 0$ when $x = 0$ implies $C = -1$. So the required particular solution can be written,

$$e^{-y} = 1 - \tan^{-1} x.$$

The question asked us to express this in the form $y = f(x)$. Taking logarithms and changing the sign on both sides gives

$$\boxed{y = -\ln\big(1 - \tan^{-1} x\big).}$$

(The domain of validity is $-\infty < x < \tan 1$.)

(b) [3 marks.] Make the substitution $u = \ln x$, $du = dx/x$. When $n \neq -1$, the integral becomes

$$\int x^{-1}(\ln x)^n\,dx = \int u^n\,du = \frac{u^{n+1}}{n+1} + C = \boxed{\frac{(\ln x)^{n+1}}{n+1} + C.}$$

(The case $n = -1$ can also be handled by the same substitution.)

(c) (i). [3 marks.] We are given

$$I_n = \int x^a (\ln x)^n\,dx, \qquad a \neq -1.$$

Apply the integration by parts formula,

$$\int UV'\,dx = UV - \int U'V\,dx, \qquad U = (\ln x)^n, \qquad V' = x^a.$$

Then $U' = (n/x)(\ln x)^{n-1}$ and $V = x^{a+1}/(a+1)$. Integration by parts gives

$$\begin{aligned} I_n &= \int x^a (\ln x)^n \, dx \\ &= \frac{x^{a+1}}{a+1}(\ln x)^n - \int \frac{x^{a+1}}{a+1} \frac{n}{x} (\ln x)^{n-1} \, dx \\ &= \frac{x^{a+1}}{a+1}(\ln x)^n - \frac{n}{a+1} \int x^a (\ln x)^{n-1} \, dx \\ &= \frac{x^{a+1}}{a+1}(\ln x)^n - \frac{n}{a+1} I_{n-1} \, . \end{aligned}$$

Hence, I_n satisfies the reduction formula,

$$\boxed{I_n = \frac{x^{a+1}}{a+1}(\ln x)^n - \frac{n}{a+1} I_{n-1} \, ,}$$

which was given on the exam paper.

(ii). [2 marks.] The given integral is I_2 for the case $a = 3$. Applying the reduction formula in part (i) twice gives

$$\begin{aligned} I_2 &= \frac{x^4}{4}(\ln x)^2 - \frac{1}{2} I_1, \\ I_1 &= \frac{x^4}{4} \ln x - \frac{1}{4} I_0, \\ I_0 &= \int x^3 \, dx = \frac{x^4}{4} + C_1. \end{aligned}$$

Putting these together and renaming the constant $C_1 = 8C$ gives

$$\begin{aligned} I_2 &= \frac{x^4}{4}(\ln x)^2 - \frac{1}{2}\left(\frac{x^4}{4} \ln x - \frac{x^4}{16} - \frac{1}{4} C_1 \right) \\ &= \frac{x^4}{32}\left\{ 8(\ln x)^2 - 4 \ln x + 1 \right\} + C. \end{aligned}$$

The required integral is

$$\boxed{\int x^3 (\ln x)^2 \, dx = \frac{x^4}{32}\left\{ 8(\ln x)^2 - 4 \ln x + 1 \right\} + C.}$$

4. (a) [6 marks: 1 for DE, 1 for general solution, 1 for C, 2 for k, 1 for total time T.] We need to choose our own notation for the physical quantities in this question. Let D denote the depth of the water in metres. (The symbol d is not recommended.) Let t denote the time in hours starting from the moment when $D = 3$. Then D is a function of t, in other words, $D = D(t)$. Torricelli's Law provides the differential equation,

$$\frac{dD}{dt} = -k\sqrt{D},$$

where k is a positive constant. We are given

$$D(0) = 3, \qquad D(24) = 2,$$

and we want to calculate the time T such that $D(T) = 0$. The differential equation is separable:

$$D^{-1/2}\,dD = -k\,dt.$$

Integrating gives

$$2\sqrt{D(t)} = -kt + C,$$

where C is a constant of integration. The initial condition $D(0) = 3$ implies $C = 2\sqrt{3}$. Hence,

$$\sqrt{D(t)} = \sqrt{3} - \tfrac{1}{2}kt.$$

The additional information $D(24) = 2$ gives $\sqrt{2} = \sqrt{3} - 12k$. This identifies

$$k = \frac{\sqrt{3} - \sqrt{2}}{12},$$

and now we have

$$\sqrt{D(t)} = \sqrt{3} - \frac{\sqrt{3} - \sqrt{2}}{24}\,t.$$

(We can, of course, get $D(t)$ explicitly by squaring both sides, but that is not necessary for this question.) Let $D(t) = 0$ when $t = T$. Then

$$\begin{aligned} T &= \frac{24\sqrt{3}}{\sqrt{3} - \sqrt{2}} \\ &= 24\sqrt{3}(\sqrt{3} + \sqrt{2}) \\ &= 24(3 + \sqrt{6}). \end{aligned}$$

Any of the last three lines is an acceptable answer to this problem. The last line is the neatest. Hence, the time taken for the tank to drain completely is

$$\boxed{T = 24(3 + \sqrt{6}) \approx 130.79 \text{ hours.}}$$

(b) [6 marks: 2 for 2nd-order DE for $x(t)$ (assuming students go this way), 1 for general solution, 2 for identifying constants, 1 for limiting temperature, final form of $x(t)$ and $y(t)$ not needed, marker's discretion for other methods.] The differential equations given for the temperature x of the hot object and the temperature y of the air in the room are

$$\frac{dx}{dt} = -\alpha(x-y), \qquad \frac{dy}{dt} = -\beta(y-x),$$

The unit of temperature is the degree Celsius and the unit of time is the hour. We are given

$$x(0) = 400, \qquad y(0) = 10, \qquad x(1) = 200, \qquad y(1) = 15.$$

There is a quick way to find the steady-state temperatures reached for large t. First, the difference $w = x - y$ satisfies the exponential decay equation $dw/dt = -(\alpha+\beta)w$, which implies that $x - y$, initially positive, decreases monotonically to zero. But then $dx/dt < 0$ and $dy/dt > 0$ for all t, and so y increases to a finite limiting temperature, say y_0, while x decreases to the same limit. This corresponds to the obvious physical fact that the object and the room will tend towards equilibrium. Dividing the two equations gives

$$\frac{dy}{dx} = -\frac{\beta}{\alpha},$$

and so $y = -(\beta/\alpha)x + K$. Putting $y = 10$ when $x = 400$ gives $K = (10\alpha + 400\beta)/\alpha$. Then putting $y = 15$ when $x = 200$ gives

$$15 = -200\frac{\beta}{\alpha} + \frac{10\alpha + 400\beta}{\alpha} = \frac{10\alpha + 200\beta}{\alpha}.$$

This identifies the following quantities:

$$\frac{\beta}{\alpha} = \frac{1}{40}, \qquad K = 20.$$

We now have the explicit relation between x and y,

$$y = 20 - \frac{x}{40},$$

valid for all $t \geq 0$ (at least while the model itself is valid, which depends on how well-insulated the room is). Now, for large t, both x and y tend towards the equilibrium temperature y_0. Then $y_0 = 20 - y_0/40$ gives the value

$$\boxed{y_0 = \frac{800}{41} \approx 19.51°\text{C}.}$$

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Of course, many students will not see the short cut and will try to derive explicit expressions for $x(t)$ and $y(t)$. Eliminating y from the pair of first-order equations gives

$$\frac{d^2x}{dt^2} + (\alpha+\beta)\frac{dx}{dt} = 0.$$

The auxiliary equation is $m(m+\alpha+\beta) = 0$. Hence, the general solution is

$$x(t) = K_1 + K_2 e^{-(\alpha+\beta)t}, \qquad y(t) = K_1 - \frac{\beta}{\alpha} K_2 e^{-(\alpha+\beta)t}.$$

The initial conditions $x(0) = 400$ and $y(0) = 10$ give

$$K_1 = \frac{10\alpha + 400\beta}{\alpha+\beta}, \qquad K_2 = \frac{390\alpha}{\alpha+\beta}.$$

The conditions $x(1) = 200$ and $y(1) = 15$ give

$$\alpha = \frac{40}{41}\ln\left(\frac{78}{37}\right), \qquad \beta = \frac{1}{41}\ln\left(\frac{78}{37}\right),$$

$$K_1 = \frac{800}{41}, \qquad K_2 = \frac{15600}{41}.$$

We could stop here since K_1 is the answer, but let us finish getting the explicit expressions,

$$x(t) = \frac{800}{41} + \frac{15600}{41}\left(\frac{37}{78}\right)^t, \qquad y(t) = \frac{800}{41} - \frac{390}{41}\left(\frac{37}{78}\right)^t.$$

As $t \to \infty$, $(37/78)^t \to 0$, and so the common limiting value of $x(t)$ and $y(t)$ is

$$y_0 = \frac{800}{41} \approx 19.51°\text{C},$$

in agreement with the faster calculation above.